MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 13: MORE ON MARKOV CHAINS. CONSTRUCTING FORECASTS WITH WEATHER.COM LIGHT

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This module uses MatLab. You want to work through it while running a MatLab session. Start one now.

We will use the material of Conversations 7, 8, 9 and Modules 11, 12.
Recall that for a Markov chain with n states and transition probability matrix $\mathbf{P}$:

- The probability distribution at time t is a row vector $\vec{\mathbf{x}}(t) = [x_1(t), \ldots, x_n(t)]$ that gives the probabilities $x_i(t)$ that the system is in state i at time t.
- The probability distribution for the next step is given by $\vec{\mathbf{x}}(t+1) = \vec{\mathbf{x}}(t)\mathbf{P}$.
- More generally, for any $k \geq 1$ and state $\vec{\mathbf{x}}(t)$, the distribution $\vec{\mathbf{x}}(t+k)$ after k time steps is given by $\vec{\mathbf{x}}(t+k) = \vec{\mathbf{x}}(t)\mathbf{P}^k$, where $\mathbf{P}^k$ is the k-step transition probability matrix.

1. Explorations of some simple Markov chains

Consider the Markov chain that was constructed in Conversation 8. Recall that state 1 means "sunny day" and state 2 means "rainy day."

First let the transition probability matrix be

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{bmatrix}$$

Create this matrix in MatLab by entering:

```
>> P = [0.4, 0.6; 0.3, 0.7]
```

Now create a state $\vec{\mathbf{x}}(t) = [0.5, 0.5]$ that corresponds to a fifty-fifty chance of rain by entering

```
>> x = [0.5, 0.5]
```

and test whether you get the same prediction for the distribution the next day as was shown on slide 6 of Conversation 9:

```
>> x*P
```

Let us also test what we get for tomorrow when we know that the current state is $\vec{\mathbf{x}}(t) = [0, 1]$ (rainy day) or $\vec{\mathbf{x}} = [1, 0]$ (sunny day):

```
>> x = [0, 1]
>> x*P
>> x = [1, 0]
>> x*P
```

You may want to verify that your answers are consistent with what was said on slide 10 of Conversation 9.

Now let us see what probability distributions we get after two days, for the day after tomorrow, for each of the distributions for today that we have explored so far. Here we want to use the formula $\vec{\mathbf{x}}(t+2) = \vec{\mathbf{x}}(t)\mathbf{P}^2$:

```
>> x = [0.5, 0.5]
>> x*P^2
>> x = [0, 1]
>> x*P^2
>> x = [1, 0]
>> x*P^2
```

The results will differ depending on the initial distribution (the one for today), but not by much.

Question 13.1: Recall that Cindy wanted to make a forecast for Saturday, based on the matrix $\mathbf{P}$ above and what happened in our story today (on Monday). What will her forecast be?

Suppose we want to trace how the forecast changes if we start with a given state $\vec{\mathbf{x}}(t)$ and then calculate forecasts $\vec{\mathbf{x}}(t+k)$ for larger and larger values of k. The easiest and most informative way to explore this in MatLab is the following:

```
>> format long
>> x = [0, 1]
>> x*P
>> ans*P
>> ans*P
```

(Repeat the last command 10 more times by using ↑ to avoid retyping it)

Now repeat the same exploration as above, but start instead with

```
>> x = [1, 0]
>> x*P
```

You will see that the predicted distributions $\vec{\mathbf{x}}(t+k)$ depend less and less on $\vec{\mathbf{x}}(t)$ as k increases. The intuitive explanation is that as we try to predict the weather in the distant future, the distribution will get closer and closer to the long-term average proportions of sunny days (state 1) and rainy days (state 2).

Now let us explore the transition probability matrix for the Markov chain with the same states that was constructed in Module 12:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 24/35 & 11/35 \\ 11/20 & 9/20 \end{bmatrix}$$

Create it in MatLab with

```
>> P = [24/35,  11/35; 11/20, 9/20]
```

Question 13.2: Pick your favorite initial state $\vec{\mathbf{x}}(t)$ and explore what happens to the distribution $\vec{\mathbf{x}}(t+k)$ as k increases. What probability of a rainy day $t+k$ does this Markov chain predict for large k? Submit your answer with an accuracy of four decimal places.

In Module 12 we constructed the above transition probability matrix from a set of observations that were recorded on 56 consecutive days, 36 of which were classified as S (sunny day, state 1), and 20 of which were classified as R (rainy day, state 2). So one might expect that the distribution $\vec{\mathbf{x}}(t+k)$ for large k should be practically the same as the vector

```
>> [36/56, 20/56]
```

But it isn't.

Question 13.3: How would you explain the observed discrepancy?

There are some transition probability matrices that allow us to make 100% accurate predictions for every future time step, at least under some conditions. For example, the following stochastic matrices have this property.

$$\mathbf{P}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{P}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{P}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{P}_4 = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

Question 13.4: Give verbal descriptions of what these matrices predict about the weather if state 1 means "sunny day" and state 2 means "rainy day." You may want to explore with MatLab what these matrices do.

2. A formal proof by mathematical induction (optional reading)

Finally, let us briefly describe how the formula $\vec{\mathbf{x}}(t+k) = \vec{\mathbf{x}}(t)\mathbf{P}^k$ follows for all positive integers k from the formula $\vec{\mathbf{x}}(t+1) = \vec{\mathbf{x}}(t)\mathbf{P}$. Note that the latter shows that $\vec{\mathbf{x}}(t+k) = \vec{\mathbf{x}}(t)\mathbf{P}^k$ holds for $k = 1$. Now assume that the formula is true for some k. We must then have $\vec{\mathbf{x}}(t+(k+1)) = \vec{\mathbf{x}}((t+k)+1)$. Taking $t+k$ as our "current time step", we know that $\vec{\mathbf{x}}((t+k)+1) = \vec{\mathbf{x}}(t+k)\mathbf{P}$, and since we assumed that the formula is true for k, we also must have $\vec{\mathbf{x}}(t+k) = \vec{\mathbf{x}}(t)\mathbf{P}^k$. By making a substitution we see then that $\vec{\mathbf{x}}(t+(k+1)) = \vec{\mathbf{x}}((t+k)+1) = \vec{\mathbf{x}}(t)\mathbf{P}^k\mathbf{P} = \mathbf{P}^{k+1}$. So, the formula is true for $k+1$.

Let us take a look at what we have shown here. The formula $\vec{\mathbf{x}}(t+k) = \vec{\mathbf{x}}(t)\mathbf{P}^k$ is true for $k = 1$. Then it must also be true for $k = 2$. Then it must also be true for $k = 3$. And so on. So we can conclude that it must be true for all positive integers k. This type of argument is called a *proof by mathematical induction.*