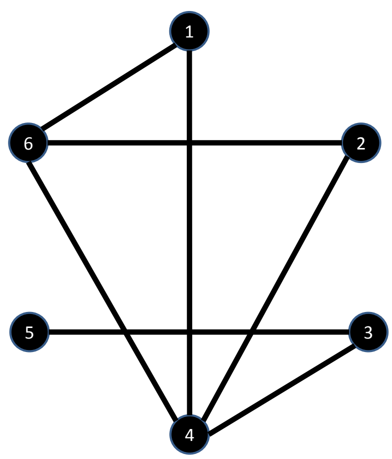
MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 14: MORE APPLICATIONS OF MARKOV CHAINS: WHERE IS WALDO?

WINFRIED JUST, OHIO UNIVERSITY

We will use the terminology and notation of Conversation 10.

Recall Waldo's studying pattern in the original version of the story: At 7p.m. he visited a randomly chosen student i among his six friends who took the same class and started working with that person. After 10 minutes, he flipped a fair coin. If the coin came up heads, he continued working with i for another 10 minutes before flipping the coin again. If the coin came up tails, he moved to the room of a randomly chosen friend of i from the same class and repeated the procedure. Here the friendships between those six classmates of Waldo are described by the following graph:



We modeled Waldo's itinerary as a Markov chain with states $1, 2, \ldots, n$ and time steps t corresponding to 10-minute intervals, and were interested in the probability distribution after 30 time steps (at midnight), which is given by the formula $\vec{\mathbf{x}}(30) = \left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right] \mathbf{P}^{30}$, where $\mathbf{P}$ denotes the matrix of transition probabilities.

Then $\vec{\mathbf{x}}(30) = [0.1427, 0.1427, 0.1432, 0.2856, 0.0717, 0.2140]$.

State 4 has largest probability here, so room 4 is were Denny should search for Waldo first. This makes sense if we think of $\vec{\mathbf{x}}(30)$ as the probabilities of the room where he is after many steps. The owner of room 4 has the largest number of friends. Whenever Waldo is in a room of one one of the friends of student number 4, that is, in room number 1, 2, 3, or 6, there is a chance that he will move next to room 4. So he will end up going to room 4 quite often. In contrast, the only place from which he can move to room 5 is from room 3, whose owner is the only friend of student number 5. Thus Waldo will spend very little time with student number 5.

Now consider the modified version of the story that Frank suggested: If Waldo is with student i, then after 10 minutes i flips a fair coin and decides whether to put up with him for another 10 minutes before flipping the coin again, or to kick him out immediately and send him to the room of a randomly chosen student who is *not* among i's friends.

We can model this scenario with another Markov chain that has the same states $1, 2, \ldots, 6$ and the same interpretation of a single time step lasting 10 minutes. However, the matrix of transition probabilities will now be different. Let's start constructing it. It will again be of order 6×6:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} \end{bmatrix}$$

Similarly, for all diagonal elements we will again have $p_{ii} = \frac{1}{2}$, since Waldo will stay in the same room for another time step whenever the coin comes up heads, and it doesn't matter here whether Waldo flips the coin or his host does:

$$\mathbf{P} = \begin{bmatrix} 1/2 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & 1/2 & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & 1/2 & p_{34} & p_{35} & p_{36} \\ p_{41} & p_{42} & p_{43} & 1/2 & p_{45} & p_{46} \\ p_{51} & p_{52} & p_{53} & p_{54} & 1/2 & p_{56} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & 1/2 \end{bmatrix}$$

Now things become different. For example, when Waldo is in room 4 and the coin comes up tails, he will be sent to room 5, because 5 is the only one among those students who is *not* a friend of 4. This allows us to fill in the fourth row of $\mathbf{P}$ as follows:

$$\mathbf{P} = \begin{bmatrix} 1/2 & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & 1/2 & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & 1/2 & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ p_{51} & p_{52} & p_{53} & p_{54} & 1/2 & p_{56} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & 1/2 \end{bmatrix}$$

Similarly, when Waldo is in one of the rooms 1, 2, 3 and the coin comes up tails (with probability 0.5), he will be sent to the room of a randomly chosen person who is *not* a friend of the owner of that room. This will be one of three persons in each case; you may want to consult the above graph of friendships about the possible choices for the next room. Since Waldo will move with probability 0.5 and there are three possibilities for the next room, the probability will be $\frac{1}{2}\frac{1}{3} = \frac{1}{6}$ for each of the possible next rooms. So we get:

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/6 & 1/6 & 0 & 1/6 & 0 \\ 1/6 & 1/2 & 1/6 & 0 & 1/6 & 0 \\ 1/6 & 1/6 & 1/2 & 0 & 0 & 1/6 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ p_{51} & p_{52} & p_{53} & p_{54} & 1/2 & p_{56} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & 1/2 \end{bmatrix}$$

Question 14.1: Complete the transition probability matrix $\mathbf{P}$ for this new Markov chain by filling in the last two rows.

Now we can get probability distribution after 30 time steps (at midnight) for this new Markov chain again from the formula $\vec{\mathbf{x}}(30) = \left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right] \mathbf{P}^{30}$, but for our new matrix $\mathbf{P}$ of transition probabilities. MatLab will give us the following answer:

$\vec{\mathbf{x}}(30) = [0.1875, 0.1875, 0.1875, 0.0625, 0.2500, 0.1250]$.

Question 14.2: Where should one search first for Waldo at midnight under this alternative version of the story?

Question 14.3: How can you intuitively explain the meaning of $\vec{\mathbf{x}}(30)$ for this new version of the story?