MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY MODULE 2: SUMMATION NOTATION

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What is $2 + 3 + \cdots + 13$?

The answer depends on what, exactly, the ellipsis $\ldots$ is supposed to stand for.

If the missing terms are supposed to be consecutive integers, it would be
$2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 = 90$.

But if the missing terms are supposed to be consecutive prime numbers, it would be
$2 + 3 + 5 + 7 + 11 + 13 = 41$.

We can see from this example that the ellipsis $\ldots$ can lead to ambiguities. It is often better to use Σ-*notation* aka *summation notation* instead. A sum of many terms can be expressed in this notation as follows:

$$a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j.$$

For example, if $a_1 = 2, a_2 = 0, a_3 = 1, a_4 = 6$, then

$\sum_{j=1}^{4} a_j = a_1 + a_2 + a_3 + a_4 = 2 + 0 + 1 + 6 = 9,$

$\sum_{j=1}^{3} a_j = a_1 + a_2 + a_3 = 2 + 0 + 1 = 3,$

$\sum_{j=1}^{1} a_j = a_1 = 2.$

Similarly, $\sum_{j=1}^{3} j^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$.

One can show that for every positive integer n the following equality holds:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}. \tag{1}$$

Note that if we multiply each of the terms a_j by the same constant c, then we get:

$$ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n),$$

$$\sum_{j=1}^{n} ca_j = c\sum_{j=1}^{n} a_j. \tag{2}$$

Practice questions

Question 2.1: Express $1 + 2 + 3 + \cdots + 33$ in summation notation and find the value of this sum.

Question 2.2: Find $\sum_{j=1}^{4} (-1)^j$.

Question 2.3: Find $\sum_{j=1}^{100} 3j$.

Question 2.4: Find a matrix $\mathbf{A} = [a_{ij}]_{3\times 3}$ with the following properties:

- $\mathbf{A}$ is symmetric, that is, $a_{ij} = a_{ji}$ for all i, j,
- $a_{11} = a_{22} = 1$,
- $a_{12} = 2a_{11}$,
- $\sum_{j=1}^{3} a_{ij} = 5$ for all i.

Question 2.5: Write down a short argument that shows that there is only one correct answer to Question 2.4.

For the following questions, assume that m is the number of students in a class, n is the number of gradable items, and the entry a_{ij} of the gradebook matrix $\mathbf{A} = [a_{ij}]_{m\times n}$ represents the score of student i on gradable item number j.

Question 2.6: Express the total score of student number i in summation notation.

Question 2.7: Express the total score of all students on gradable item number j in summation notation.

Question 2.8: Express the mean score of all students on gradable item number j in summation notation.

Question 2.9: Express in summation notation Bob's method of calculating the mean total score: Add up the total scores of all students and divide the sum by m.

Question 2.10: Express in summation notation Denny's method of calculating the mean total score: Add up the mean scores of all gradable items.

If you answered these questions correctly and factor out a constant as in Equation (2) above, you will see that the two methods result in adding up the same terms, only in a different order.