MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 25: ELEMENTARY ROW OPERATIONS AND ELEMENTARY MATRICES

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This module uses MatLab. You want to work through it while running a MatLab session. Start one now.

This module builds on the material of Conversation 13.

Recall from Conversation 13 that when we apply one of the following operations to a system of linear equations, we end up with an equivalent system, that is, with a system that has the same solution set:

(i) Interchanging the positions of any two equations.

(ii) Multiplying an equation by a nonzero scalar.

(iii) Adding to one equation a scalar multiple of another equation.

Let us consider an example and see what these operations do to the extended, or augmented matrix of a system.

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & = & 1 \\ -x_1 & & & + & x_3 & = & 2 \end{array} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & 1 & -1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}$$

Switching the order of the first two equations corresponds to switching the first two rows of the extended matrix:

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & + & 3x_2 & - & 2x_3 & = & 0 \\ -x_1 & & & + & x_3 & = & 2 \end{array} \qquad [\mathbf{A}_1, \vec{\mathbf{b}}_1] = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 3 & -2 & 0 \\ -1 & 0 & 1 & 2 \end{bmatrix}$$

Multiplying the third equation of the new system by 4 corresponds to multiplying the third row of the new extended matrix by 4:

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & + & 3x_2 & - & 2x_3 & = & 0 \\ -4x_1 & & & + & 4x_3 & = & 8 \end{array} \qquad [\mathbf{A}_2, \vec{\mathbf{b}}_2] = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 3 & -2 & 0 \\ -4 & 0 & 4 & 8 \end{bmatrix}$$

Adding 2 times the first equation of the new system to its third equation corresponds to adding 2 times the first row of the new extended matrix to its third row:

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & + & 3x_2 & - & 2x_3 & = & 0 \\ & & 2x_2 & + & 2x_3 & = & 10 \end{array} \qquad [\mathbf{A}_3, \vec{\mathbf{b}}_3] = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 3 & -2 & 0 \\ 0 & 2 & 2 & 10 \end{bmatrix}$$

These examples illustrate that when we apply any of the following *elementary row operations* to the extended matrix of a system of linear equations, we end up with an *equivalent matrix,* that is with the extended matrix of an *equivalent linear system,* that is, of a linear system that has the same solution set as the original one:

(E1) Interchanging any two rows.

(E2) Multiplying any row by a nonzero scalar.

(E3) Adding to one row of the matrix a scalar times another row of the matrix.

In this module we will illustrate that each of the elementary row operations can be implemented by multiplying matrices with so-called *elementary matrices.*
First let us create the matrix $[\mathbf{A}, \vec{\mathbf{b}}]$ above:

```
>> Ab = [1,3,-2,0;2,1,-1,1;-1,0,1,2]
```

Now let us create three elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ and then explore what they do:

```
>> E1 =  [0,1,0;1,0,0;0,0,1]
>> E2 =  [1,0,0;0,1,0;0,0,4]
>> E3 =  [1,0,0;0,1,0;2,0,1]
>> B = E1*Ab
>> B = E2*ans
>> B = E3*ans
```

You will see that we obtain, successively, the matrices $[\mathbf{A}_1, \vec{\mathbf{b}}_1]$, $[\mathbf{A}_2, \vec{\mathbf{b}}_2]$, $[\mathbf{A}_3, \vec{\mathbf{b}}_3]$ that we had created above by applying elementary row operations.

Question 25.1: Suppose that you want to find a matrix $\mathbf{E}$ that implements these three successive operations in the sense that $\mathbf{E}[\mathbf{A}, \vec{\mathbf{b}}] = [\mathbf{A}_3, \vec{\mathbf{b}}_3]$. How would this matrix be related to $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$?

Now let us explore three more elementary matrices:

```
>> E4 =  [0,0,1;0,1,0;1,0,0]
>> E5 =  [1,0,0;0,-1,0;0,0,1]
>> E6 =  [1,5,0;0,1,0;0,0,1]
```

Question 25.2: Enter `>> B  = E4*Ab`. What does multiplication from the left with $\mathbf{E}_4$ do to $[\mathbf{A}, \vec{\mathbf{b}}]$?

Question 25.3: Enter `>> B  = E5*Ab`. What does multiplication from the left with $\mathbf{E}_5$ do to $[\mathbf{A}, \vec{\mathbf{b}}]$?

Question 25.4: Enter `>> B  = E6*Ab`. What does multiplication from the left with $\mathbf{E}_6$ do to $[\mathbf{A}, \vec{\mathbf{b}}]$?