MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 26: GAUSSIAN ELIMINATION, PART I

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Question 26.1: Solve the following linear system using Gaussian elimination:

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 0 \\ 4x_1 & + & 5x_2 & = & 6 \end{array}$$

Question 26.2: Solve the following linear system using Gaussian elimination:

$$\begin{array}{rcrcrcr} -x_1 & + & 2x_2 & + & x_3 & = & 1 \\ x_1 & - & x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & - & x_3 & = & 3 \end{array}$$

Question 26.3: Solve the following linear system using Gaussian elimination:

$$\begin{array}{rcrcrcrcr} -x_1 & + & 2x_2 & + & x_3 & + & x_4 & = & 1 \\ x_1 & - & x_2 & + & x_3 & + & x_4 & = & 2 \\ x_1 & + & x_2 & - & x_3 & + & x_4 & = & 3 \\ x_1 & + & x_2 & + & x_3 & - & x_4 & = & 4 \end{array}$$

Question 26.4: Solve the following linear system using Gaussian elimination:

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 4x_3 & = & 5 \\ 4x_1 & + & 6x_2 & + & 3x_3 & = & 0 \\ 2x_1 & + & 3x_2 & - & x_3 & = & -5 \end{array}$$

Question 26.5: Solve the following linear system using Gaussian elimination:

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 4x_3 & = & 5 \\ 3x_1 & + & 2x_2 & - & x_3 & = & 0 \\ x_1 & + & 5x_2 & - & 5x_3 & = & 5 \end{array}$$

Question 26.6: Find the mistake in the following description of solving the system:

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & -4 \\ 3x_1 & + & 4x_2 & = & 7 \end{array}$$

Step 1: The augmented matrix is $[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & -4 \\ 3 & 4 & 7 \end{bmatrix}$

Now we perform Gaussian elimination on the augmented matrix:

Step 2: $\begin{bmatrix} 1 & 2 & -4 \\ 3 & 4 & 7 \end{bmatrix} \xrightarrow{R2-3R1} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -2 & 19 \end{bmatrix}$

Step 3: $\begin{bmatrix} 1 & 2 & -4 \\ 0 & -2 & 19 \end{bmatrix} \xrightarrow{R2/(-2)} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -19/2 \end{bmatrix}$

Step 4: This matrix is in row echelon form and is the extended matrix of the equivalent system:

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & -4 \\ & & x_2 & = & \frac{-19}{2} \end{array}$$

Step 5: We can read off x_2 immediately, and then use back-substitution to find $x_1 = 15$.

Step 6: Thus the solution of the system is $x_1 = 15$.

Question 26.7: Find the mistake in the following description of solving the system:

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 4x_3 & = & 5 \\ 4x_1 & + & 3x_2 & + & 6x_3 & = & 0 \\ 6x_1 & - & x_2 & + & 2x_3 & = & 5 \end{array}$$

Step 1: The augmented matrix is $[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 6 & 3 & 0 \\ 6 & -1 & 2 & 5 \end{bmatrix}$

Perform Gaussian elimination:

Step 2: $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 6 & 3 & 0 \\ 6 & -1 & 2 & 5 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 2R1} \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 0 & -5 & -10 \\ 6 & -1 & 2 & 5 \end{bmatrix}$

Step 3: $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 0 & -5 & -10 \\ 6 & -1 & 2 & 5 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - 3R1} \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 0 & -5 & -10 \\ 0 & -10 & -10 & -10 \end{bmatrix}$

Step 4: $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 0 & -5 & -10 \\ 0 & -10 & -10 & -10 \end{bmatrix} \xrightarrow{R3 \leftrightarrow R2} \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -10 & -10 & -10 \\ 0 & 0 & -5 & -10 \end{bmatrix}$

Step 5: $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -10 & -10 & -10 \\ 0 & 0 & -5 & -10 \end{bmatrix} \xrightarrow{R1 \mapsto R1/2} \begin{bmatrix} 1 & 1.5 & 2 & 2.5 \\ 0 & -10 & -10 & -10 \\ 0 & 0 & -5 & -10 \end{bmatrix}$

Step 6: $\begin{bmatrix} 1 & 1.5 & 2 & 2.5 \\ 0 & -10 & -10 & -10 \\ 0 & 0 & -5 & -10 \end{bmatrix} \xrightarrow{R2 \mapsto R2/(-10)} \begin{bmatrix} 1 & 1.5 & 2 & 2.5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & -10 \end{bmatrix}$

Step 7: $\begin{bmatrix} 1 & 1.5 & 2 & 2.5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & -10 \end{bmatrix} \xrightarrow{R3 \mapsto R3/(-5)} \begin{bmatrix} 1 & 1.5 & 2 & 2.5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

Step 8: We have transformed the augmented matrix of the original system into an equivalent matrix in row echelon form that represents the following equivalent system:

$$\begin{array}{rcrcrcr} x_1 & + & 1.5x_2 & + & 2x_3 & = & 2.5 \\ & & x_2 & + & x_3 & = & 1 \\ & & & & x_3 & = & 2 \end{array}$$

Step 9: Back-substitution gives the solution as the vector with coordinates: $x_3 = 2, \quad x_2 = -1, \quad x_1 = 0.$