MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 27: GAUSSIAN ELIMINATION, PART II

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Question 27.1: Solve the following linear system using Gaussian elimination:

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 4x_3 & = & 5 \\ 3x_1 & & & - & x_3 & = & 0 \\ 5x_1 & - & 3x_2 & + & 3x_3 & = & 5 \end{array}$$

Question 27.2: Solve the following linear system using Gaussian elimination:

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 4x_3 & = & 5 \\ 4x_1 & + & 6x_2 & + & 3x_3 & = & 0 \\ 6x_1 & - & x_2 & + & 2x_3 & = & 5 \end{array}$$

Question 27.3: Solve the following linear system by Gaussian elimination:

$$\begin{array}{rcrcr} 2x_1 & - & 3x_2 & = & 5 \\ 6x_1 & - & 9x_2 & = & 10 \end{array}$$

Question 27.4: Solve the following linear system by Gaussian elimination:

$$\begin{array}{rcrcrcr} 2x_1 & - & 4x_2 & + & 8x_3 & = & -2 \\ 2x_1 & + & 6x_2 & + & 3x_3 & = & 6 \\ 6x_1 & - & 2x_2 & + & 27x_3 & = & 2 \end{array}$$

Question 27.5: Solve the following linear system by Gaussian elimination:

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 3x_2 & + & 4x_3 & & & = & 5 \\ 3x_1 & & & - & x_3 & + & x_4 & = & 6 \\ 5x_1 & - & 3x_2 & + & 3x_3 & + & x_4 & = & 7 \end{array}$$

Question 27.6: Find the mistake in the following description of solving the system:

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 4x_3 & = & 5 \\ 3x_1 & + & 2x_2 & - & x_3 & = & 0 \\ x_1 & + & 5x_2 & - & 5x_3 & = & 5 \end{array}$$

Step 1: The augmented matrix is $[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 2 & -3 & 4 & 5 \\ 3 & 2 & -1 & 0 \\ 1 & 5 & -5 & 5 \end{bmatrix}$

Perform Gaussian elimination:

Step 2: $\begin{bmatrix} 2 & -3 & 4 & 5 \\ 3 & 2 & -1 & 0 \\ 1 & 5 & -5 & 5 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R3} \begin{bmatrix} 1 & 5 & -5 & 5 \\ 3 & 2 & -1 & 0 \\ 2 & -3 & 4 & 5 \end{bmatrix}$

Step 3: $\begin{bmatrix} 1 & 5 & -5 & 5 \\ 3 & 2 & -1 & 0 \\ 2 & -3 & 4 & 5 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 3R1} \begin{bmatrix} 1 & 5 & -5 & 5 \\ 0 & -13 & 14 & -15 \\ 2 & -3 & 4 & 5 \end{bmatrix}$

Step 4: $\begin{bmatrix} 1 & 5 & -5 & 5 \\ 0 & -13 & 14 & -15 \\ 2 & -3 & 4 & 5 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - 2R1} \begin{bmatrix} 1 & 5 & -5 & 5 \\ 0 & -13 & 14 & -15 \\ 0 & -13 & 14 & -5 \end{bmatrix}$

Step 5: $\begin{bmatrix} 1 & 5 & -5 & 5 \\ 0 & -13 & 14 & -15 \\ 0 & -13 & 14 & -5 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R2} \begin{bmatrix} 1 & 5 & -5 & 5 \\ 0 & -13 & 14 & -15 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

Step 6: $\begin{bmatrix} 1 & 5 & -5 & 5 \\ 0 & -13 & 14 & -15 \\ 0 & -13 & 14 & -5 \end{bmatrix} \xrightarrow{C3 \mapsto C3 + C4} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -13 & -1 & -15 \\ 0 & 0 & 10 & 10 \end{bmatrix}$

Step 7: $\begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -13 & -1 & -15 \\ 0 & 0 & 10 & 10 \end{bmatrix} \xrightarrow{R2 \mapsto R2/(-13)} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & 1 & 1/13 & 15/13 \\ 0 & 0 & 10 & 10 \end{bmatrix}$

Step 8: $\begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & 1 & 1/13 & 15/13 \\ 0 & 0 & 10 & 10 \end{bmatrix} \xrightarrow{R3 \mapsto R2/10} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & 1 & 1/13 & 15/13 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Step 9: The resulting matrix in row echelon form represents the system

$$\begin{array}{rcrcrcr} x_1 & + & 5x_2 & & & = & 5 \\ & & x_2 & + & \frac{1}{13}x_3 & = & \frac{15}{13} \\ & & & & x_3 & = & 1 \end{array}$$

Step 10: By back-substitution we find that the solution is the vector with coordinates $x_3 = 1$, $x_2 = \frac{14}{13}$, and $x_1 = \frac{-5}{13}$.

Question 27.7: Find the mistake in the following description of solving the system:

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 4x_3 & = & 5 \\ 3x_1 & & & - & x_3 & = & 0 \\ 5x_1 & - & 3x_2 & + & 3x_3 & = & 5 \end{array}$$

Step 1: The augmented matrix is $[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 2 & -3 & 4 & 5 \\ 3 & 0 & -1 & 0 \\ 5 & -3 & 3 & 5 \end{bmatrix}$

Perform Gaussian elimination:

Step 2: $\begin{bmatrix} 2 & -3 & 4 & 5 \\ 3 & 0 & -1 & 0 \\ 5 & -3 & 3 & 5 \end{bmatrix} \xrightarrow{R1 \mapsto 0.5R1} \begin{bmatrix} 1 & -1.5 & 2 & 2.5 \\ 3 & 0 & -1 & 0 \\ 5 & -3 & 3 & 5 \end{bmatrix}$

Step 3: $\begin{bmatrix} 1 & -1.5 & 2 & 2.5 \\ 3 & 0 & -1 & 0 \\ 5 & -3 & 3 & 5 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 3R1} \begin{bmatrix} 1 & -1.5 & 2 & 2.5 \\ 0 & 4.5 & -7 & -7.5 \\ 5 & -3 & 3 & 5 \end{bmatrix}$

Step 4: $\begin{bmatrix} 1 & -1.5 & 2 & 2.5 \\ 0 & 4.5 & -7 & -7.5 \\ 5 & -3 & 3 & 5 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - 5R1} \begin{bmatrix} 1 & -1.5 & 2 & 2.5 \\ 0 & 4.5 & -7 & -7.5 \\ 0 & 4.5 & -7 & -7.5 \end{bmatrix}$

Step 5: $\begin{bmatrix} 1 & -1.5 & 2 & 2.5 \\ 0 & 4.5 & -7 & -7.5 \\ 0 & 4.5 & -7 & -7.5 \end{bmatrix} \xrightarrow{R3 \mapsto R3 - R2} \begin{bmatrix} 1 & -1.5 & 2 & 2.5 \\ 0 & 4.5 & -7 & -7.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Step 6: $\begin{bmatrix} 1 & -1.5 & 2 & 2.5 \\ 0 & 4.5 & -7 & -7.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \mapsto R2/4.5} \begin{bmatrix} 1 & -1.5 & 2 & 2.5 \\ 0 & 1 & -14/9 & -15/9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Step 7: We have transformed the augmented matrix of the original system into an equivalent matrix in row echelon form that represents the following equivalent system:

$$\begin{array}{rcrcrcr} x_1 & - & 1.5x_2 & + & 2x_3 & = & 2.5 \\ & & x_2 & - & \frac{14}{9}x_3 & = & -\frac{15}{9} \\ & & & & 0 & = & 0 \end{array}$$

Step 8: Since the last equation does not make sense, this system is inconsistent.