MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 28: INTRODUCTION TO INVERSE MATRICES

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Let $\mathbf{A}$ be an $n \times n$ square matrix. Recall from Lecture 16 that the *inverse of* $\mathbf{A}$ is an $n \times n$ matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$. Moreover, recall that *if the inverse* $\mathbf{A}^{-1}$ *exists,* it is unique and satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$.

Note that *only* square matrices can have inverses, but not all square matrices do.

In order to *verify* that two matrices are inverses of each other, we simply need to multiply these matrices and check whether the product is the identity matrix of the relevant order.

Question 28.1: Determine which among the following matrices are inverse to each other:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0.5 & -1.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0.5 \\ 2 & -1.5 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0.6 & 0.8 \\ 0.2 & -0.4 \end{bmatrix}$$

Finding inverse matrices is in general not easy; we will learn a method for this in later lectures. In some special cases though it is easy; you have already seen in Lecture 16 how this works for diagonal matrices.

Question 28.2: Which of the following matrices is/are invertible?
What are the inverse matrices if they exist?

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Other matrices whose inverses are relatively easy to find are the elementary matrices that we studied in Module 25. Recall that when $\mathbf{E}$ is an elementary matrix, then the corresponding elementary row operation on a matrix $\mathbf{A}$ can be performed by multiplying $\mathbf{A}$ from the left with $\mathbf{E}$. Then $\mathbf{E}^{-1}(\mathbf{E}\mathbf{A}) = (\mathbf{E}^{-1}\mathbf{E})\mathbf{A} = \mathbf{I}\mathbf{A} = \mathbf{A}$, so $\mathbf{E}^{-1}$ *undoes* the elementary row operation that is implemented by $\mathbf{E}$. Thus in order to find $\mathbf{E}^{-1}$, one can think about what it would take to undo the corresponding elementary row operation. This process will also boil down to applying an elementary row operation, and its elementary matrix is the inverse matrix to the given one.

For example, consider the following matrix:

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

This is the elementary matrix for switching rows 2 and 5 of any $5 \times n$ matrix $\mathbf{A}$.

Question 28.3: What would $\mathbf{E}^{-1}$ be? *Hint:* Consider what you need to do in order to undo the elementary row operation described above.

Not all square matrices have inverses. In particular, no square zero matrix $\mathbf{O}_{n\times n}$ with all entries zero has an inverse. We have also seen other examples in Lecture 16. There are many more such examples.

Question 28.4: If $\mathbf{A}$ is any 2×2 matrix that has two zeros in its second column, then $\mathbf{A}$ does not have an inverse. Consider the following "Proof" of this fact and identify the step that is wrong. How would you fix this proof?

"Proof:" *Step 1:* Let $\mathbf{A}$ is any 2×2 matrix that has two zeros in its second column.

Step 2: We can write $\mathbf{A}$ in the form $\mathbf{A} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$

Step 3: Consider the product of $\mathbf{A}$ with an arbitrary 2×2 matrix $\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

Step 4: Then $\mathbf{AC} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} ac_{11} + 0c_{21} & 0c_{12} + 0c_{22} \\ bc_{11} + 0c_{21} & 0c_{12} + 0c_{22} \end{bmatrix}$

Step 5: $\begin{bmatrix} ac_{11} + 0c_{21} & 0c_{12} + 0c_{22} \\ bc_{11} + 0c_{21} & 0c_{12} + 0c_{22} \end{bmatrix} = \begin{bmatrix} ac_{11} & 0 \\ bc_{11} & 0 \end{bmatrix}$

Step 6: $\begin{bmatrix} ac_{11} & 0 \\ bc_{11} & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ because the elements in the lower right corner are different.

Step 7: Therefore no 2×2 matrix $\mathbf{C}$ can be the inverse matrix of $\mathbf{A}$. $\square$

Question 28.5: If $\mathbf{A}$ is any 2×2 matrix such that its second row is a scalar multiple of the first, then $\mathbf{A}$ does not have an inverse. Consider the following "Proof" of this fact and identify the step that is wrong. How would you fix this proof?

"Proof:" *Step 1:* Let $\mathbf{A}$ is any 2×2 matrix such that its second row is a scalar multiple of the first.

Step 2: We can assume that the second row is two times the first row and write $\mathbf{A}$ in the form

$\mathbf{A} = \begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}$

Step 3: Consider the product of $\mathbf{A}$ with an arbitrary 2×2 matrix $\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

Step 4: Then $\mathbf{AC} = \begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} ac_{11} + bc_{21} & ac_{12} + bc_{22} \\ 2ac_{11} + 2bc_{21} & 2ac_{12} + 2bc_{22} \end{bmatrix}$

Step 5: We can see that the second row of this product is 2 times the first row.

Step 6: $\begin{bmatrix} ac_{11} + bc_{21} & ac_{12} + bc_{22} \\ 2ac_{11} + 2bc_{21} & 2ac_{12} + 2bc_{22} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

because in the identity matrix on the right the second row is not 2 times the first row.

Step 7: Therefore no 2×2 matrix $\mathbf{C}$ can be the inverse matrix of $\mathbf{A}$. $\square$