MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 29: USING THE INVERSE OF THE COEFFICIENT MATRIX TO SOLVE LINEAR SYSTEMS

WINFRIED JUST, OHIO UNIVERSITY

This module is based on the material of Lecture 17. Recall the following facts from this lecture:

Consider a linear equation $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where $\mathbf{A}$ is square.

- If $\mathbf{A}$ *is invertible,* then $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$ is the unique solution.
- If $\mathbf{A}$ *is non-invertible,* then the system is either underdetermined or inconsistent.

The first observation allows us to find solutions of linear systems $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ as matrix products $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$, as long as the coefficient matrix $\mathbf{A}$ is invertible and its inverse is known. This is especially useful when we want to solve several systems with the same coefficient matrix.

In Lecture 16 we found that the inverse matrix of $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

Question 29.1: Find the solutions $\vec{\mathbf{x}}$ of the following linear systems using the matrix product $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$:

System A:
$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 2 \\ 3x_1 & + & 4x_2 & = & -1 \end{array}$$

System B:
$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 1 \\ 3x_1 & + & 4x_2 & = & 3 \end{array}$$

System C:
$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 0 \\ 3x_1 & + & 4x_2 & = & 1 \end{array}$$

Consider the matrices $\mathbf{A} = \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -1 & 0.5 \\ 0.5 & 0.5 & -0.25 \\ -0.5 & 0.5 & 0.25 \end{bmatrix}$

Question 29.2: Verify that $\mathbf{A}^{-1} = \mathbf{B}$.

Question 29.3: Find the solutions of the following linear systems using the matrix product $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$:

System A:
$$\begin{array}{rcrcrcr} 0.5x_1 & + & x_2 & & & = & -1 \\ & & x_2 & + & x_3 & = & 2 \\ x_1 & & + & & 2x_3 & = & 3 \end{array}$$

System B:
$$\begin{array}{rcrcrcr} 0.5x_1 & + & x_2 & & & = & 1 \\ & & x_2 & + & x_3 & = & -3 \\ x_1 & & + & & 2x_3 & = & 2 \end{array}$$

System C:
$$\begin{array}{rcrcrcr} 0.5x_1 & + & x_2 & & & = & 3 \\ & & x_2 & + & x_3 & = & -2 \\ x_1 & & + & & 2x_3 & = & 1 \end{array}$$

Question 29.4: Consider the following argument. Is it correct? If not, what is wrong with it?

Let us solve the system of linear equations:
$$\begin{array}{rcrcr} 2x_1 & + & 5x_2 & = & 4 \\ x_1 & + & 3x_2 & = & -1 \end{array}$$

The coefficient matrix is $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Let $\mathbf{B} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ Then $\mathbf{BA} = \mathbf{I}$, so that $\mathbf{B} = \mathbf{A}^{-1}$.

Form the extended matrix $\begin{bmatrix} 3 & -5 & 4 \\ -1 & 2 & -1 \end{bmatrix}$ and perform Gaussian elimination:

$$\begin{bmatrix} 3 & -5 & 4 \\ -1 & 2 & -1 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} -1 & 2 & -1 \\ 3 & -5 & 4 \end{bmatrix} \xrightarrow{R1 \mapsto -R1} \begin{bmatrix} 1 & -2 & 1 \\ 3 & -5 & 4 \end{bmatrix} \xrightarrow{R2 \mapsto R2 - 3R1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

We obtained the extended matrix of the equivalent system
$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & 1 \\ & & x_2 & = & 1 \end{array}$$

By back-substitution we find that the unique solution is the vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$