MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 30: GAUSS-JORDAN ELIMINATION

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This module is based on Lecture 18. Recall from this lecture that Gauss-Jordan elimination works as follows:

Let $\mathbf{A}$ be an $n \times n$ matrix. Form an $n \times 2n$ matrix $\mathbf{C}$ by dropping the internal brackets in $[\mathbf{A}, \mathbf{I}_n]$ and replacing them with a vertical dividing line for visual clarity. For $n = 3$ we get:

$$\mathbf{C} = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Perform Gaussian elimination on $\mathbf{C}$. If *the first half* of the resulting matrix in generalized row echelon form has a zero row, then $\mathbf{A}$ is not invertible. *Otherwise* keep going and apply instances of (E3) until the first half turn into $\mathbf{I}_n$ and we get a matrix in reduced row echelon form.

For $n = 3$ the result will look like:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right]$$

The matrix $\mathbf{B}$ *in the second half will be the inverse* $\mathbf{A}^{-1}$.

Question 30.1: Perform Gauss-Jordan elimination for the following matrix. Determine whether the inverse matrix exist, and if it does, find it.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$$

Question 30.2: Perform Gauss-Jordan elimination for the following matrix. Determine whether the inverse matrix exist, and if it does, find it.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & -3 \\ 3 & 1 & 4 \end{bmatrix}$$

Question 30.3: Perform Gauss-Jordan elimination for the following matrix. Determine whether the inverse matrix exist, and if it does, find it.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & -3 \\ 6 & 6 & 6 \end{bmatrix}$$

Question 30.4: Perform Gauss-Jordan elimination for the following matrix. Determine whether the inverse matrix exist, and if it does, find it.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & -1 & -2 \\ 2 & 3 & 4 \end{bmatrix}$$

Question 30.5: Let be a matrix of the form $\mathbf{A} = \begin{bmatrix} 1 & a & 0 \\ 0 & -1 & b \\ 0 & 0 & 1 \end{bmatrix}$

By performing Gauss-Jordan elimination, find a formula for $\mathbf{A}^{-1}$.

Question 30.6: Consider the following argument. Is it correct? If not, find all mistakes in it.

Let us calcuate $\mathbf{A}^{-1}$ for $\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 3 & -2 \\ -0.5 & -5 & 1 \end{bmatrix}$

Step 1: Form a 3×6 matrix $\mathbf{C} = \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 3 & -2 & 0 & 1 & 0 \\ -0.5 & -5 & 1 & 0 & 0 & 1 \end{array}\right]$

Perform Gaussian elimination on $\mathbf{C}$:

Step 2: $\left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 3 & -2 & 0 & 1 & 0 \\ -0.5 & -5 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.5 & -5 & 1 & 0 & 0 & 1 \end{array}\right]$

Step 3: $\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.5 & -5 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3 \mapsto R3 + 0.5R1} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -3.5 & 0 & 0 & 0.5 & 1 \end{array}\right]$

Step 4: $\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -3.5 & 0 & 0 & 0.5 & 1 \end{array}\right] \xrightarrow{R3 \mapsto R3 - 3.5R2} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3.5 & 0.5 & 1 \end{array}\right]$

Step 5: $\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3.5 & 0.5 & 1 \end{array}\right] \xrightarrow{R2 \mapsto -R2} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3.5 & 0.5 & 1 \end{array}\right]$

We still need to cancel the off-diagonal element in the first half of the matrix:

Step 6: $\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3.5 & 0.5 & 1 \end{array}\right] \xrightarrow{R1 \mapsto R1 - 3R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3.5 & 0.5 & 1 \end{array}\right]$

Step 7: Since all rows of the resulting 3×6 matrix are nonzero, we can conclude that the matrix in the second half is $\mathbf{A}^{-1} = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 0 & 0 \\ -3.5 & 0.5 & 1 \end{bmatrix}$