MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 31: PROPERTIES OF INVERSE MATRICES

WINFRIED JUST, OHIO UNIVERSITY

This module is based on Lecture 19. Recall the following properties of matrix inverses from this lecture:

- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
- $\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$.
- The inverse of an invertible upper-triangular matrix is also upper-triangular.
- The inverse of an invertible lower-triangular matrix is also lower-triangular.

Question 31.1: Is the following argument correct? If not, which step(s) are/is wrong?

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ $\quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad \mathbf{C} = \begin{bmatrix} 10 & 3 \\ 4 & 2 \end{bmatrix}$ Consider the following argument:

Step 1: Since $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1.5 \\ 0 & 0.5 \end{bmatrix} = \mathbf{I}$, we have $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1.5 \\ 0 & 0.5 \end{bmatrix}$

Step 2: Since $\begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.25 & 0 \\ -0.5 & 1 \end{bmatrix} = \mathbf{I}$, we have $\mathbf{B}^{-1} = \begin{bmatrix} 0.25 & 0 \\ -0.5 & 1 \end{bmatrix}$

Step 3: $\mathbf{C} = \mathbf{AB}$.

Step 4: Thus $\mathbf{C}^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1} = \begin{bmatrix} 1 & -1.5 \\ -0.25 & 0.5 \end{bmatrix}$

Question 31.2: Prove that the inverse of every invertible symmetric matrix is symmetric.
Hint: Recall that a matrix $\mathbf{A}$ is symmetric if, and only if, $\mathbf{A} = \mathbf{A}^T$. Then start by expressing the property you want to prove in terms of a matrix transpose, and use a property that you have seen in Lecture 19.

Let us now illustrate the proof that the inverse matrix of an invertible upper-triangular matrix is upper-triangular. More specifically, we will prove that the inverse matrix of an invertible 3×3 upper triangular matrix

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \text{ is an upper-triangular matrix } \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$$

Let $\mathbf{U}$ be as above. We form:

$$[\mathbf{U}, \mathbf{I}_3] = \left[\begin{array}{ccc|ccc} u_{11} & u_{12} & u_{13} & 1 & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 & 1 & 0 \\ 0 & 0 & u_{33} & 0 & 0 & 1 \end{array}\right]$$

In Chapter 4 we will prove that and upper triangular matrix $\mathbf{U}$ is invertible if, and only if, all elements u_{ii} on the (main) diagonal are nonzero. Since we are assuming here that $\mathbf{U}$ is invertible, this allows us to divide the first row by $u_{11} \neq 0$:

$$\left[\begin{array}{ccc|ccc} u_{11} & u_{12} & u_{13} & 1 & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 & 1 & 0 \\ 0 & 0 & u_{33} & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & u'_{12} & u'_{13} & b_{11} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 & 1 & 0 \\ 0 & 0 & u_{33} & 0 & 0 & 1 \end{array}\right]$$

Divide the second row by $u_{22} \neq 0$:

$$\left[\begin{array}{ccc|ccc} 1 & u'_{12} & u'_{13} & b_{11} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 & 1 & 0 \\ 0 & 0 & u_{33} & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & u'_{12} & u'_{13} & b_{11} & 0 & 0 \\ 0 & 1 & u'_{23} & 0 & b_{22} & 0 \\ 0 & 0 & u_{33} & 0 & 0 & 1 \end{array}\right]$$

Divide the third row by $u_{33} \neq 0$:

$$\left[\begin{array}{ccc|ccc} 1 & u'_{12} & u'_{13} & b_{11} & 0 & 0 \\ 0 & 1 & u'_{23} & 0 & b_{22} & 0 \\ 0 & 0 & u_{33} & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & u'_{12} & u'_{13} & b_{11} & 0 & 0 \\ 0 & 1 & u'_{23} & 0 & b_{22} & 0 \\ 0 & 0 & 1 & 0 & 0 & b_{33} \end{array}\right]$$

Use the pivot 1 in the lower right corner to cancel u'_{23}:

$$\left[\begin{array}{ccc|ccc} 1 & u'_{12} & u'_{13} & b_{11} & 0 & 0 \\ 0 & 1 & u'_{23} & 0 & b_{22} & 0 \\ 0 & 0 & 1 & 0 & 0 & b_{33} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & u'_{12} & u'_{13} & b_{11} & 0 & 0 \\ 0 & 1 & 0 & 0 & b_{22} & b_{23} \\ 0 & 0 & 1 & 0 & 0 & b_{33} \end{array}\right]$$

Use the pivot 1 in the lower right corner to cancel u'_{13}:

$$\left[\begin{array}{ccc|ccc} 1 & u'_{12} & u'_{13} & b_{11} & 0 & 0 \\ 0 & 1 & 0 & 0 & b_{22} & b_{23} \\ 0 & 0 & 1 & 0 & 0 & b_{33} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & u'_{12} & 0 & b_{11} & 0 & b'_{13} \\ 0 & 1 & 0 & 0 & b_{22} & b_{23} \\ 0 & 0 & 1 & 0 & 0 & b_{33} \end{array}\right]$$

Use the pivot 1 in the second row to cancel u'_{12}:

$$\left[\begin{array}{ccc|ccc} 1 & u'_{12} & 0 & b_{11} & 0 & b'_{13} \\ 0 & 1 & 0 & 0 & b_{22} & b_{23} \\ 0 & 0 & 1 & 0 & 0 & b_{33} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & 0 & b_{22} & b_{23} \\ 0 & 0 & 1 & 0 & 0 & b_{33} \end{array}\right]$$

The elements b_{ij} on the right are left here in symbolic form; but it is not too hard to figure out formulas for them.

Question 31.3: By tracing the steps in this derivation, find expressions for $b_{11}, b_{12}, b_{13}, b_{22}, b_{23}, b_{33}$ in terms of $u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}$.