MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 4: ADJACENCY MATRICES AND DEGREE SEQUENCES OF UNDIRECTED GRAPHS

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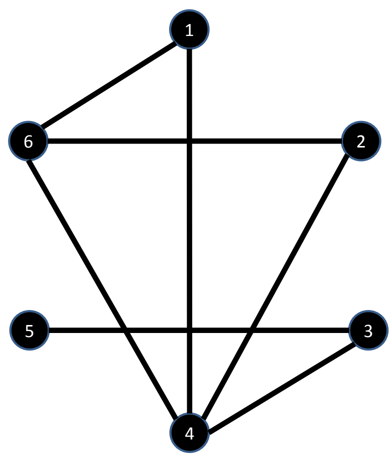
We will use here the notation and terminology of Lecture 3.

1. Self-study: Degrees of nodes in graphs and degree sequences

Consider the graph G of friendships from Lecture 3:



When you look at the graph, you will see that nodes 1, 2, 3 are connected by 2 edges each to other nodes, node 4 is connected by 4 edges to other nodes, node 5 by only one such edge, and node 6 is connected by 3 such edges. These numbers of edges are called the *degrees* of the nodes, and the degree of a particular node i is denoted by $deg(i)$.

Question 4.1: In our interpretation of the graph G as representing friendships, what does $deg(i)$ tell us about node i?

We can arrange the degrees into a row vector as follows:

$\vec{\mathbf{d}} = [deg(1), deg(2), deg(3), deg(4), deg(5), deg(6)] = [2, 2, 2, 4, 1, 3]$.

We will call $\vec{\mathbf{d}}$ the *degree sequence* of the graph G.

Alternatively, we could define the degree sequence as a column vector, but it will be more convenient to work with row vectors here.

It is interesting to see how the degree sequence relates to the adjacency matrix of a graph. For the graph G in our example, we found that its adjacency matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} = [a_{ij}]_{6\times 6}$$

When you sum up each column, you get the vector $[2, 2, 2, 4, 1, 3]$, which is exactly the degree sequence! It is not hard to understand why this must always be the case: There will be exactly as many ones in column number i as there are edges in the graph that connect with node i. For the same reason, had we decided to represent our degree sequence as a column vector, we would obtain it by summing up the rows of the adjacency matrix.

2. Practice on adjacency matrices and degree sequences

Consider graphs that represents friendships among a group of people numbered 1 through 4.

Question 4.2: Draw this friendship graph when the friends of 1 are 2 and 3, the only friend of 4 is 3, and there are no other friendships than the ones mentioned.

Question 4.3: Find the adjacency matrix $\mathbf{A}$ for the graph that you drew for Question 4.2.

Question 4.4: Find the degree sequence for the graph that you drew for Question 4.2.

Question 4.5: Form a matrix $\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} - \mathbf{A}$ by subtracting your adjacency matrix $\mathbf{A}$.

Notice that $\mathbf{B}$ is also an adjacency matrix of a graph. Draw this graph.

Question 4.6: Find the degree sequence for the graph that you drew for Question 4.5.

Question 4.7: What do the edges of the graph with adjacency matrix $\mathbf{B}$ represent?

Consider the following adjacency matrix $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} = [a_{ij}]_{m\times n}$

Question 4.8: Find the degree sequence for the graph with adjacency matrix $\mathbf{C}$.

Question 4.9: Draw the graph with adjacency matrix $\mathbf{C}$.