MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 42: FINDING COEFFICIENTS OF LINEAR COMBINATIONS

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This module is based on Lecture 22.

Recall from Lecture 22 the following method for determining whether a given vector $\vec{\mathbf{w}}$ is a linear combination of given vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ and finding coefficients if it is.

- Form a matrix $\mathbf{A}$ that either has $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ (if they are column vectors) or $\vec{\mathbf{v}}_1^T, \vec{\mathbf{v}}_2^T, \dots, \vec{\mathbf{v}}_n^T$ as its successive columns.
- Let $\vec{\mathbf{b}}$ be either $\vec{\mathbf{w}}$ or $\vec{\mathbf{w}}^T$, depending on whether $\vec{\mathbf{w}}$ is a column or a row vector.
- Solve the linear system with extended matrix $[\mathbf{A}, \vec{\mathbf{b}}]$.
- If this system is *inconsistent,* $\vec{\mathbf{w}}$ is *not* a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$.
- If this system is *consistent,* $\vec{\mathbf{w}}$ *is* a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ and *each* solution vector will give you coefficients of a linear combination.

We also observed that any system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is consistent if, and only if, the vector $\vec{\mathbf{b}}$ is a linear combination of the column vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_n$ of its coefficient matrix $\mathbf{A}$.

Question 42.1: Consider the statement:

"The linear system $\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & & x_2 & - & x_3 & = & 4 \\ & & & & x_3 & = & 5 \end{array}$ is consistent."

If you want to rephrase this as: "The vector $\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$ is a linear combination of vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3$," what should these vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3$ be?

Question 42.2: Is the statement that you rephrased in Question 42.1 actually true? If so, find suitable coefficients for the linear combination.

Question 42.3: Consider the following vectors:

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \qquad \vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix} \qquad \vec{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 7 \\ 6 \end{bmatrix} \qquad \vec{\mathbf{w}} = \begin{bmatrix} -3 \\ 10 \\ 51 \end{bmatrix}$$

Is $\vec{\mathbf{w}}$ a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$? If so, find coefficients for the linear combination.

Question 42.4: Consider the following vectors:

$$\vec{\mathbf{v}}_1 = [1, -2, 4], \qquad \vec{\mathbf{v}}_2 = [6, 18, 10], \qquad \vec{\mathbf{v}}_3 = [-2, -11, -1], \qquad \vec{\mathbf{w}} = [0, -7, -6].$$

Is $\vec{\mathbf{w}}$ a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$? If so, find coefficients for the linear combination.

Question 42.5: Consider the following vectors:

$$\vec{\mathbf{v}}_1 = [1, -2, 4], \qquad \vec{\mathbf{v}}_2 = [6, 18, 10], \qquad \vec{\mathbf{v}}_3 = [-2, -11, -1], \qquad \vec{\mathbf{w}} = [10, 10, 26].$$

Is $\vec{\mathbf{w}}$ a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$? If so, find coefficients for the linear combination.