MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 46: VECTOR SPACES AND THEIR SPANNING SETS

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This module is based on Lecture 23. Recall the following definitions from this Lecture:

- We call a subset V of some $\mathbb{R}^n$ that is of the form $V = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$ for some subset $S = \{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k\}$ of $\mathbb{R}^n$ a *linear subspace of* $\mathbb{R}^n$. In this course, the phrase *vector space* will always mean "a linear subspace of $\mathbb{R}^n$ for some positive integer n."
- The set S is then called *a spanning set of* V and we will sometimes write $V = span(S)$ instead of $V = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$.

Notice the indefinite article "a" in the definition of "a spanning set." This is because a given vector space V will usually have infinitely many different spanning sets. Consider, for example, the vectors $\vec{\mathbf{v}}_1 = [1, 0, 0], \vec{\mathbf{v}}_2 = [0, 1, 0], \vec{\mathbf{v}}_3 = [0, 0.5, 0]$ in $\mathbb{R}^3$. Let $V = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$. This is the x-y-plane, since by definition $\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ if, and only if,
$\vec{\mathbf{w}} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2 = x\,[1, 0, 0] + y\,[0, 1, 0] = [x, y, 0]$ for some coefficients x, y.
Similarly, $\vec{\mathbf{w}}$ is a linear combination of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_3$ if, and only if,
$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_3\vec{\mathbf{v}}_3 = d_1\,[1, 0, 0] + d_3\,[0, 0.5, 0] = [d_1, 0.5d_3, 0]$ for some coefficients d_1, d_3.
By choosing $d_1 = x$ and $d_3 = 2y$, we can obtain any given vector $[x, y, 0]$ in the x-y-plane as a linear combination of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_3$, which shows that $V = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_3)$.
Every vector $\vec{\mathbf{w}} = [x, y, 0]$ in the x-y-plane V can also be written as the linear combination
$\vec{\mathbf{w}} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2 + 0\vec{\mathbf{v}}_3 = x\,[1, 0, 0] + y\,[0, 1, 0] + 0\,[0, 0.5, 0] = [x, y, 0]$.
Conversely, every linear combination of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ is in the x-y-plane:
$d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + d_3\vec{\mathbf{v}}_3 = d_1\,[1, 0, 0] + d_2\,[0, 1, 0] + d_3\,[0, 0.5, 0] = [d_1, d_2 + 0.5d_3, 0]$.
Thus $V = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)$.

Notice that in the last example the vector $\vec{\mathbf{v}}_3$ was *already* in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$, and then $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)$ turned out to be equal to $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$. This will happen in general; let us state this important observation as a formal mathematical result and try to prove it.

Proposition 1. *Consider a set* $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1}\}$ *of vectors of the same order.*

(a) *Every vector* $\vec{\mathbf{w}}$ *in* $span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$ *is also in* $span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1})$.
(b) *If* $\vec{\mathbf{v}}_{k+1}$ *is in* $span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$, *then* $span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k) = span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1})$.

Proof: Part (a): If $\vec{\mathbf{w}}$ is in $span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$, then $\vec{\mathbf{w}}$ is a linear combination
$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + \cdots + d_k\vec{\mathbf{v}}_k$ for some coefficients $d_1, \ldots, d_k$.
But then for the same $d_1, \ldots, d_k$ we get
$\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + \cdots + d_k\vec{\mathbf{v}}_k + 0\vec{\mathbf{v}}_{k+1}$, so $\vec{\mathbf{w}}$ is in $span(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1})$.

For the proof of part (b) it now suffices to show that if a vector $\vec{\mathbf{w}}$ is in $span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1})$, then $\vec{\mathbf{w}}$ is already in $span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$. We will give here two versions of this argument:

Version 1: Assume that $\vec{\mathbf{v}}_{k+1}$ is in $span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$, and let $\vec{\mathbf{w}}$ be in $\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1}\}$. Then there exist coefficients $c_1, \dots, c_k$ and $d_1, \dots, d_k, d_{k+1}$ such that

$\vec{\mathbf{v}}_{k+1} = c_1\vec{\mathbf{v}}_1 + \cdots + c_k\vec{\mathbf{v}}_k$ and $\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + \cdots + d_k\vec{\mathbf{v}}_k + d_{k+1}\vec{\mathbf{v}}_{k+1}$.

By substituting the above expression for $\vec{\mathbf{v}}_{k+1}$ into the above expression for $\vec{\mathbf{w}}$ we find that

$$\begin{aligned}\vec{\mathbf{w}} &= d_1\vec{\mathbf{v}}_1 + \cdots + d_k\vec{\mathbf{v}}_k + d_{k+1}\vec{\mathbf{v}}_{k+1}\\ &= d_1\vec{\mathbf{v}}_1 + \cdots + d_k\vec{\mathbf{v}}_k + d_{k+1}(c_1\vec{\mathbf{v}}_1 + \cdots + c_k\vec{\mathbf{v}}_k)\\ &= (d_1 + d_{k+1}c_1)\vec{\mathbf{v}}_1 + \cdots + (d_k + d_{k+1}c_k)\vec{\mathbf{v}}_k.\end{aligned}$$

This shows that $\vec{\mathbf{w}}$ is already a linear combination of $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$, with coefficients $d_1 + d_{k+1}c_1, \dots, d_k + d_{k+1}c_k$.

Version 2: Assume that $\vec{\mathbf{v}}_{k+1}$ is in $span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$, and let $\vec{\mathbf{w}}$ be in $span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1})$. Then each of the vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1}$ is in $span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$. This $\vec{\mathbf{w}}$ is a linear combination of vectors in $span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$, and since the linear span of any sets of vectors is closed under linear combinations (see the theorem on slide 7 of Lecture 23), $\vec{\mathbf{w}}$ must also be in $span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$.

Question 46.1: Which version(s) of the proof of part (b) is (are) correct?

Question 46.2: Consider the set V of all solutions of the linear equation $x_1 + x_2 = 0$. This set is a line through the origin, hence a vector space. Which of the following are spanning sets of V?

$$S_1 = \left\{\begin{bmatrix}2\\-2\end{bmatrix}\right\} \quad S_2 = \left\{\begin{bmatrix}20\\20\end{bmatrix}\right\} \quad S_3 = \left\{\begin{bmatrix}2\\-2\end{bmatrix}, \begin{bmatrix}20\\20\end{bmatrix}\right\} \quad S_4 = \left\{\begin{bmatrix}2\\-2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\} \quad S_5 = \left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$$

Question 46.3: Consider the following vectors:

$\vec{\mathbf{v}}_1 = [1, 2, 3]$, $\vec{\mathbf{v}}_2 = [0, 0, 3]$, $\vec{\mathbf{v}}_3 = [1, 0, 1]$, $\vec{\mathbf{v}}_4 = [2, 0, 1]$

(a) Can you express $\vec{\mathbf{v}}_1$ as a linear combination of $\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4$?
(b) Can you express $\vec{\mathbf{v}}_4$ as a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$?
(c) What do points (a) and (b) imply about the relationship between $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4)$, $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)$, and $span(\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4)$? Which of these sets are equal?