MATH3200: APPLIED LINEAR ALGEBRA PRACTICE MODULE 47A: DEFINITIONS OF LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

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This module is based on Conversation 25.

1. Definitions of linear dependence and linear independence

Recall the following definitions from Conversation 25:

Definition 1 (Tentative Definition). *A set* $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ *of vectors of the same order is* linearly dependent *if, and only if, one of these vectors can be expressed as a linear combination of the other vectors.*

This set is linearly independent *if, and only if, it is not linearly dependent.*

Definition 2 (Official Definition). *A set* $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ *of vectors of the same order is* linearly dependent *if, and only if, there are scalars* $c_1, c_2, \dots, c_k$*, not all of them zero, so that*
$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.

This set is linearly independent *if, and only if, it is not linearly dependent.*

In Conversation 25 we illustrated that these definitions are equivalent for every set of $k > 1$ vectors; here we will formally prove this fact. In Lecture 24 it will be shown that these definitions are also equivalent when we only have $k = 1$ vector in our set.

Let $k > 1$ and let $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ be a set of vectors that is linearly dependent in the sense of the tentative definition. Then one of these vectors is a linear combination of the other vectors. We don't know which one that is, but since linear dependence and linear independence are properties of *sets* of vectors, not of individual vectors, we can change the numbering of our set linearly dependent S in such a way that a vector that is a linear combination of the other vectors is listed first, as $\vec{\mathbf{v}}_1$. Then there are coefficients $d_2, \dots, d_k$ such that

$\vec{\mathbf{v}}_1 = d_2\vec{\mathbf{v}}_2 + \cdots + d_k\vec{\mathbf{v}}_k$ and after subtracting $\vec{\mathbf{v}}_1$ from both sides of the equation we obtain
$\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_1 = \vec{\mathbf{0}} = -\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_k\vec{\mathbf{v}}_k$.

Question 47.1: In the line that immediately precedes this question, the expression on the right gives coefficients for a linear combination so that
$\vec{\mathbf{0}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$.
What can you say about these coefficients $c_1, c_2, \dots, c_k$?

Your answer to Question 47.1 will show that not all of the coefficients $c_1, c_2, \dots, c_k$ can be zero, so that the set $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ is linearly dependent in the sense of the official definition.

For the proof in the other direction, assume that the set $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ is linearly dependent in the sense of the official definition. Then there are coefficients $c_1, c_2, \dots, c_k$ such that $\vec{\mathbf{0}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$, and least one of these coefficients is different from zero. Again we don't know which coefficient that is, but we can again change the numbering of the vectors in S if need be so that $c_1 \neq 0$. Mathematicians refer to this trick of "changing the numbering if need be" by using the phrase *we may without loss of generality assume that* $c_1 \neq 0$.

Question 47.2: Under the assumption that $c_1 \neq 0$, how would you express $\vec{\mathbf{v}}_1$ as a linear combination of the other vectors in the set S?

Your answer to Question 47.2 will show that one of the vectors in the set S, more specifically, $\vec{\mathbf{v}}_1$ under our assumption about c_1, is a linear combination of the other vectors in this set, so that S is linearly dependent in the sense of our tentative definition.