MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 47B: MORE ON LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

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This module is based on Conversation 25 and Lecture 24.

1. Review: Definitions and a characterization of linear dependence and linear independence

Recall the following definitions and Theorems from Conversation 25 and Lecture 24:

Definition 1 (Tentative Definition)**.** *A set* $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ *of vectors of the same order is* linearly dependent *if, and only if, one of these vectors can be expressed as a linear combination of the other vectors.*

This set is linearly independent *if, and only if, it is not linearly dependent.*

Definition 2 (Official Definition)**.** *A set* $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ *of vectors of the same order is* linearly dependent *if, and only if, there are scalars* $c_1, c_2, \dots, c_k$*, not all of them zero, so that*
$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.

This set is linearly independent *if, and only if, it is not linearly dependent.*

We have proved in Lecture 24 and Module47A that these definitions are equivalent. In particular, either of these definitions tells us that a set $\{\vec{\mathbf{v}}_1\}$ that contains exactly one vector is linearly dependent if $\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$ and linearly independent if $\vec{\mathbf{v}}_1 \neq \vec{\mathbf{0}}$.

Theorem 1. *Suppose* $S^- = \{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k\}$ *and* $S^+ = \{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_{k+1}, \dots, \vec{\mathbf{v}}_\ell\}$ *are sets of vectors that are all of the same order such that every vector in* S^- *is also in* S^+*. This can also be expressed by writing that* S^- *is a* subset *of* S^+ *and* S^+ *is a* superset *of* S^-*. Then*

- *If* S^- *is linearly dependent, then* S^+ *is also linearly dependent.*
- *If* S^+ *is linearly independent, then* S^- *is also linearly independent.*

Theorem 2. *Let* $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ *be a set of vectors of the same order. Then these vectors are linearly independent if, and only if,* every *vector* $\vec{\mathbf{w}}$ *in span*$(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k)$ *can be expressed as*
$\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$
for exactly one *choice of the coefficients* $c_1, c_2, \dots, c_n$.

2. Some properties of linearly (in)dependent sets

Question 47.3: Suppose $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$ is a set of vectors in $\mathbb{R}^5$ such that the set
$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4, [4, -8, 36, 44, -48]\}$ is linearly independent.
Is S linearly dependent or linearly independent?

Question 47.4: Consider the following set of vectors:
$S = \{[0, 1, 2, 3, 4], [3, 6, 8, 12, 17], [0, 2, 4, 6, 8], [1, 3, 7, 5, 9], [-4, 8, 23, -29, 48]\}$.
Is S linearly dependent or linearly independent?

Question 47.5: Consider the following set of vectors:
$S = \{[0,1,2,3,4], [3,6,8,12,17], [1,3,7,5,9], [0,0,0,0,0], [-4,8,23,-29,48]\}$.
Is S linearly dependent or linearly independent?

Question 47.6: Should the empty set of vectors be considered linearly dependent or linearly independent?

Question 47.7: Suppose a system of linear equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is underdetermined. Does this tell us anything about linear dependence or linear independence of the set of column vectors of $\mathbf{A}$? *Hint:* Use Theorem 2.

3. A procedure for determining whether a given set of vectors is linearly dependent or linearly independent

It is important for us to have a method for recognizing linear (in)dependence. Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ be a set of vectors of the same order. We are looking for coefficients $c_1, c_2, \dots, c_k$ such that
$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.
As we learned in Lecture 22 and Module 42, this boils down to solving a homogeneous system of linear equations $\mathbf{A}\vec{\mathbf{c}} = \vec{\mathbf{0}}$, where the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k$, or their transposes $\vec{\mathbf{v}}_1^T, \vec{\mathbf{v}}_2^T, \dots, \vec{\mathbf{v}}_k^T$, are written as the columns of $\mathbf{A}$ and $\vec{\mathbf{c}} = [c_1, c_2, \dots, c_k]^T$.
If the system is *underdetermined* the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ are *linearly dependent;* otherwise they are *linearly independent.*

Consider the following example that was discussed in Conversation 25:
Let $\vec{\mathbf{v}}_1 = [2,-1,3]$, $\vec{\mathbf{v}}_2 = [3,4,5]$, $\vec{\mathbf{v}}_3 = [6,-7,8]$.
In order to determine whether the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is linearly dependent of linearly independent, we want to find all vectors of coefficients c_1, c_2, c_3 with
$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 = c_1[2,-1,3] + c_2[3,4,5] + c_3[6,-7,8] = \vec{\mathbf{0}}$.
These must be the solutions of the linear system

$$\begin{array}{rcrcrcl} 2c_1 & + & 3c_2 & + & 6c_3 & = & 0 \\ -c_1 & + & 4c_2 & - & 7c_3 & = & 0 \\ 3c_1 & + & 5c_2 & + & 8c_3 & = & 0 \end{array}$$

We can do this by performing a Gaussian elimination on the extended matrix $[\mathbf{A}, \vec{\mathbf{b}}]$ of this system:

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 2 & 3 & 6 & 0 \\ -1 & 4 & -7 & 0 \\ 3 & 5 & 8 & 0 \end{bmatrix} \xrightarrow{R1 \mapsto R1/2} \begin{bmatrix} 1 & 1.5 & 3 & 0 \\ -1 & 4 & -7 & 0 \\ 3 & 5 & 8 & 0 \end{bmatrix} \xrightarrow{R2 \mapsto R2+R1} \begin{bmatrix} 1 & 1.5 & 3 & 0 \\ 0 & 5.5 & -4 & 0 \\ 3 & 5 & 8 & 0 \end{bmatrix} \xrightarrow{R3 \mapsto R3-3R1}$$

$$\begin{bmatrix} 1 & 1.5 & 3 & 0 \\ 0 & 5.5 & -4 & 0 \\ 0 & 0.5 & -1 & 0 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 1.5 & 3 & 0 \\ 0 & 0.5 & -1 & 0 \\ 0 & 5.5 & -4 & 0 \end{bmatrix} \xrightarrow{R3 \mapsto R3-11R2} \begin{bmatrix} 1 & 1.5 & 3 & 0 \\ 0 & 0.5 & -1 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} \xrightarrow{R2 \mapsto 2R2}$$

$$\begin{bmatrix} 1 & 1.5 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} \xrightarrow{R3 \leftrightarrow R3/7} \begin{bmatrix} 1 & 1.5 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We obtained a matrix in row echelon form that is the extended matrix of the equivalent system

$$\begin{array}{ccccccc} c_1 & + & 1.5c_2 & + & 3c_3 & = & 0 \\ & & c_2 & - & 2c_3 & = & 0 \\ & & & & c_3 & = & 0 \end{array}$$

Back-substitution shows that the only solution satisfies $c_1 = c_2 = c_3 = 0$. Thus we conclude that the zero vector $\vec{\mathbf{0}}$ can be obtained as a linear combination
$c_1[2, -1, 3] + c_2[3, 4, 5] + c_3[6, -7, 8] = \vec{\mathbf{0}}$
only when all coefficients c_1, c_2, c_3 are equal to zero, and we conclude that the set of vectors $\{[2, -1, 3], [3, 4, 5], [6, -7, 8]\}$ is linearly independent.

Question 47.8: Determine whether the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} = \{[1, 2, 3], [-3, 20, 17], [-5, 3, -2]\}$ is linearly dependent or linearly independent.

Question 47.9: Determine whether the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} = \{[1, 2, 3], [0, 0, 3], [1, 0, 1]\}$ is linearly dependent or linearly independent.