MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 48: BASES OF VECTOR SPACES

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This module is based on Conversation 26 and Lecture 25A,B and references Modules 46 and 47, as well as Lecture 18.

Recall the following definitions and facts:

Let V be a vector space. A set $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ such that $V = span(S) = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ is called a *spanning set* of V.

A linearly independent spanning set of V is called a *basis* of V.

We can always obtain a basis of vector space $V = span(S) = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ by successively removing vectors from S that are linear combinations of the other vectors until we are left with a linearly independent subset of V. However, usually not all bases of V can be obtained in this way.

Every two bases of V have the same size. This size $dim(V)$ is called the *dimension* of V.

Every vector space other than the zero-dimensional space $\{\vec{\mathbf{0}}\}$ has infinitely many different bases.

For a given n, we let $\vec{\mathbf{e}}_i$ denote the vector in $\mathbb{R}^n$ that has 1 in position i and 0 in all other positions. The set $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n\}$ forms the *standard basis* of $\mathbb{R}^n$. Its elements $\vec{\mathbf{e}}_i$ are called *standard basis vectors.*

Given any basis $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ of a vector space V, for every vector $\vec{\mathbf{w}}$ in V there exists exactly one vector $\vec{\mathbf{c}} = [c_1, \ldots, c_k]^T$ of coefficients such that $\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$.

These vectors $\vec{\mathbf{c}}$ give us *coordinates* for the elements of V and can be used to *parametrize* V.

When $V = \mathbb{R}^n$ and B is the standard basis, we get the *Cartesian coordinates*; for other bases B we get *alternative coordinates* with respect to B.

Suppose $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is a basis of a linear subspace V of some $\mathbb{R}^n$.
Let $\mathbf{B}$ be a matrix whose columns contain these basis vectors as columns in the given order, written in Cartesian coordinates.
Consider a vector $\vec{\mathbf{w}}$ in V with Cartesian coordinates $\vec{\mathbf{x}}$ and alternative coordinates $\vec{\mathbf{c}} = [c_1, \ldots, c_k]^T$ with respect to B. Then

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{x}} \tag{1}$$

Thus we can compute $\vec{\mathbf{x}}$ from $\vec{\mathbf{c}}$ as the matrix product

$$\vec{\mathbf{x}} = \mathbf{B}\vec{\mathbf{c}} \tag{2}$$

We can find the $\vec{\mathbf{c}}$ by solving the above system (1) of linear equations.

When $k = n$ and B is a basis for the entire space, then $\mathbf{B}$ is invertible and we can compute $\vec{\mathbf{c}}$ from $\vec{\mathbf{x}}$ as the matrix product

$$\vec{\mathbf{c}} = \mathbf{B}^{-1}\vec{\mathbf{x}} \tag{3}$$

Question 48.1: Consider the set $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$, where:
$\vec{\mathbf{v}}_1 = [1, 2, 3],\ \vec{\mathbf{v}}_2 = [0, 0, 3],\ \vec{\mathbf{v}}_3 = [1, 0, 1],\ \vec{\mathbf{v}}_4 = [2, 0, 1]$

(a) Find a basis of V that is a subset of S.
(b) What is the dimension $dim(V)$ of the vector space V?
(c) Find all bases for V that are subsets of the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$.
(d) Find another basis for V that does not contain any vector from the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$.

Hint: Think about which vectors you can and which vectors you cannot remove from S while preserving the linear span. Use your answers to Questions 46.3 and 47.8 of Modules 46 and 47.

Now consider, for example, the following sets of vectors in $\mathbb{R}^3$:
$B_1 = \{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3\} = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$ and $B_2 = \{[1, 1, 1], [2, 3, 4], [-5, 1, 0]\}$.
Both are bases of $\mathbb{R}^3$. For example, the vector $\vec{\mathbf{w}} = [-9, 6, -10]$ can be expressed as a linear combination of the vectors in B_1 as
$\vec{\mathbf{w}} = [-9, 6, -10] = x_1[1, 0, 0] + x_2[0, 1, 0] + x_3[0, 0, 1] = -9[1, 0, 0] + 6[0, 1, 0] - 10[0, 0, 1]$
with coefficients $x_1 = -9, x_2 = 6, x_3 = -10$, and there are no other choices for the coefficients. Since B_1 is the standard basis for $\mathbb{R}^3$, these are the *Cartesian coordinates* of the vector $\vec{\mathbf{w}}$.
Similarly, the same vector $\vec{\mathbf{w}}$ can be expressed as a linear combination of the vectors in B_2 as
$\vec{\mathbf{w}} = [-9, 6, -6] = c_1[1, 1, 1] + c_2[2, 3, 4] + c_3[-5, 1, 0] = 2[1, 1, 1] - 3[2, 3, 4] + [-5, 1, 0]$
with coefficients $c_1 = 2, c_2 = -3, c_3 = 1$, and there are no other choices for these coefficients. Here c_1, c_2, c_3 would be the *alternative coordinates* of $\vec{\mathbf{w}}$ with respect to B_2.

Now let us try to find the alternative coordinates c_1, c_2, c_3 with respect to B_2 of the vector with Cartesian coordinates $x_1 = -8, x_2 = 23, x_3 = 26$. Here we need coefficients c_1, c_2, c_3 such that $[-8, 23, 26] = c_1[1, 1, 1] + c_2[2, 3, 4] + c_3[-5, 1, 0]$.
We can find these coefficients by solving the linear system

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & - & 5c_3 & = & -8 \\ c_1 & + & 3c_2 & + & c_3 & = & 23 \\ c_1 & + & 4c_2 & & & = & 26 \end{array} \tag{4}$$

We could use Gaussian elimination to solve this system. We could also form the matrix

$\mathbf{B} = \begin{bmatrix} 1 & 2 & -5 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$ that contains the transposes of the vectors in B_2 as its columns,

and use Equation (3): $\vec{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{B}^{-1}\vec{\mathbf{x}} = \mathbf{B}^{-1} \begin{bmatrix} -8 \\ 23 \\ 26 \end{bmatrix}$

As $\mathbf{B}$ is the coefficient matrix of the system (4), both methods must give the same result.
The second method requires knowledge of $\mathbf{B}^{-1}$, but if we do happen to know it, it is faster.

Here it will be given to you: $\mathbf{B}^{-1} = \begin{bmatrix} 4/7 & 20/7 & -17/7 \\ -1/7 & -5/7 & 6/7 \\ -1/7 & 2/7 & -1/7 \end{bmatrix}$

Question 48.2: Find the alternative coordinates c_1, c_2, c_3 with respect to B_2 of the vector with Cartesian coordinates $x_1 = -8, x_2 = 23, x_3 = 26$.

Question 48.3: Find the Cartesian coordinates x_1, x_2, x_3 of the vector with alternative coordinates $c_1 = 7, c_2 = -7, c_3 = 21$ with respect to B_2.

Let us remark here that while Equation (2) always applies when B is any basis of any vector space; Equation (3) can be used only if V is the entire space $\mathbb{R}^n$. In cases where we want to parametrize a low-dimensional subspace of $\mathbb{R}^n$ for some large n with alternative coordinates, these coordinates still need to be found by solving the relevant systems of linear equation with some method other than using Equation (3).

For the next two question though these formulas work and the relevant inverse matrices were computed in Lecture 18.

Question 48.4: Consider the following basis of $\mathbb{R}^2$: $B = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$

(a) Express the vector $\vec{\mathbf{x}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in alternative coordinates $\vec{\mathbf{c}}$ with respect to B.

(b) Find the Cartesian coordinates $\vec{\mathbf{x}}$ of the vector $\vec{\mathbf{c}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ that is written in alternative coordinates with respect to B.

Question 48.5: Consider the following basis of $\mathbb{R}^3$: $B = \left\{ \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix} \right\}$

(a) Express the vector $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in alternative coordinates $\vec{\mathbf{c}}$ with respect to B.

(b) Find the Cartesian coordinates $\vec{\mathbf{x}}$ of the vector $\vec{\mathbf{c}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ that is written in alternative coordinates with respect to B.