MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 49: THE RANK OF A MATRIX

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This module is based on Lecture 26.

1. The row space and the column space of a matrix

The *row space* $RS(\mathbf{A}) = span(\vec{\mathbf{a}}_{1*}, \vec{\mathbf{a}}_{2*}, \dots, \vec{\mathbf{a}}_{m*})$ of a matrix $\mathbf{A}$ is the linear span of all of its rows.
The *column space* $CS(\mathbf{A}) = span(\vec{\mathbf{a}}_{*1}, \vec{\mathbf{a}}_{*2}, \dots, \vec{\mathbf{a}}_{*n})$ of a matrix $\mathbf{A}$ is the linear span of all of its columns.
$RS(\mathbf{A})$ and $CS(\mathbf{A})$ are different vector spaces, but must have the same dimension $dim(RS(\mathbf{A})) = dim(CS(\mathbf{A})) = r(\mathbf{A})$, called the *rank* of $\mathbf{A}$.
The literature uses a variety of different symbols for the row space and the column space of a matrix; here we will stick to the notation that was defined above.

Question 49.1: Let $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ Give verbal descriptions of the row space $RS(\mathbf{A})$ and the column space $CS(\mathbf{A})$.

Question 49.2: Let $\mathbf{A} = \begin{bmatrix} 2 & 5 & -3 \\ 8 & 8 & 6 \end{bmatrix}$ Give verbal descriptions of the row space $RS(\mathbf{A})$ and the column space $CS(\mathbf{A})$.

2. Computing the rank of a matrix

The rank $r(\mathbf{A})$ is equal to the maximum size of a linearly independent subset of its rows, aka the *row rank* of $\mathbf{A}$, and is also equal to the the maximum size of a linearly independent subset of its columns, aka the *column rank* of $\mathbf{A}$.
The rank of a row-reduced matrix is equal to the number of its nonzero rows, and is also equal to the number of its *pivotal columns,* that is, columns that contain a first nonzero element of some row. For example, in the following matrix in row-echelon form, the pivotal columns will be columns 1,3, and 4. Note that there are exactly as many such rows as there are nonzero rows; this will be true for every matrix in generalized row-echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since Gaussian elimination preserves the rank of a matrix, we can compute the rank of any given matrix $\mathbf{A}$ as follows:

- Step 1: Perform the steps of Gaussian elimination on $\mathbf{A}$ until you obtain a matrix in generalized row-echelon form.

- Step 2: Count the number of pivotal columns or the number of nonzero rows. This will be the rank.

Recall the following example from Lecture 26 that illustrates this procedure.

Consider the following matrix $\mathbf{A}$ and start Gaussian elimination on it:

$$\begin{bmatrix} 0 & 14 & -83 & 22 & -7 \\ 0 & 0 & 0 & 31 & -25 \\ 0 & 28 & -166 & 13 & 11 \end{bmatrix} \xrightarrow{R3 \mapsto R3-2R1} \begin{bmatrix} 0 & 14 & -83 & 22 & -7 \\ 0 & 0 & 0 & 31 & -25 \\ 0 & 0 & 0 & -31 & 25 \end{bmatrix} \xrightarrow{R3 \mapsto R3+R2} \begin{bmatrix} 0 & 14 & -83 & 22 & -7 \\ 0 & 0 & 0 & 31 & -25 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have obtained a matrix in generalized row-echelon form. It is not necessary to divide rows by their first nonzero elements, as we can already see that there are two pivotal columns; column number 2 and column number 4. Thus we can conclude that the rank of this matrix and of our original one is $r(\mathbf{A}) = 2$.

Question 49.3: Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix}$

Question 49.4: Find the rank of the matrix $\mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 6 & 5 & 6 \end{bmatrix}$

Question 49.5: Find the rank of the matrix $\mathbf{C} = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 5 & 0 & 6 \\ -1 & 1 & -1 & 1 \\ 5 & 6 & 2 & 5 \end{bmatrix}$

Recall that an $n \times n$ square matrix $\mathbf{A}$ is said to have *full rank* if $r(\mathbf{A}) = n$, that is, if its column vectors (equivalently: its row vectors) form a linearly independent set.

Question 49.6: Which of the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ of Questions 49.3–49.5 have (has) full rank?