MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 52: THE NULL SPACE OF A MATRIX

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This module is based on Lecture 28.

Consider a system of linear equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ with coefficient matrix $\mathbf{A}$ of order $m \times n$. Recall from Lecture 28 that

- The *null space* $N(\mathbf{A})$ of $\mathbf{A}$ is the set of column vectors $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ such that $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
- The dimension of the null space is given by $dim(N(\mathbf{A})) = n - r(\mathbf{A})$.
- In order to find a basis for the null space, we find the solution set, we can proceed as follows:
  (1) Solve the homogenous system of linear equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
  (2) If there is a unique solution, then $N(\mathbf{A}) = \{\vec{\mathbf{0}}\}$ and the empty set is the basis.
  (3) Otherwise the solution set can be described in terms of choosing $k = dim(N(\mathbf{A}))$ among the variables as *free parameters.* We obtain a set of basis vectors for $N(\mathbf{A})$ by successively setting each of these free parameters to 1 while setting all other free parameters to 0.

Question 52.1: Prove that the null space of an $m \times n$ matrix $\mathbf{A}$ is a linear subspace of $\mathbb{R}^n$. *Hint:* Recall from Lecture 23 that it suffices to show that if $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ are in $N(\mathbf{A})$ and if λ is a scalar, then $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ and $\lambda\vec{\mathbf{x}}$ are also in $N(\mathbf{A})$.

Question 52.2: Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$ Find $r(\mathbf{A}), dim(N(\mathbf{A}))$, and $N(\mathbf{A})$.

Question 52.3: Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$ Find a basis of $N(\mathbf{A})$.

Question 52.4: Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 1 \\ 3 & -6 & 4 \end{bmatrix}$ Find $r(\mathbf{A}), dim(N(\mathbf{A}))$, and $N(\mathbf{A})$.

Question 52.5: Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 1 \\ 3 & -6 & 4 \end{bmatrix}$ Find a basis of $N(\mathbf{A})$.

Question 52.6: Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -1 & 2 & -3 \end{bmatrix}$ Find $r(\mathbf{A}), dim(N(\mathbf{A}))$, and $N(\mathbf{A})$.

Question 52.7: Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -1 & 2 & -3 \end{bmatrix}$ Find a basis of $N(\mathbf{A})$.