MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 53: THE RANK OF THE COEFFICIENT MATRIX AND THEORY OF SOLUTIONS OF A LINEAR SYSTEM, PART I: THREE THEOREMS

WINFRIED JUST, OHIO UNIVERSITY

This module is based on Conversation 29 and Lecture 29.

Let us explicitly state all three theorems that were discussed in this conversation:

Theorem 1. *Consider a linear system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *with coefficient matrix* $\mathbf{A}$ *of order* $m \times n$*.*

- *When* $r(\mathbf{A}) = m$*, the system is always consistent.*
- *When* $r(\mathbf{A}) < m$*, the system is consistent for some, but not for all choices of* $\vec{\mathbf{b}}$*.*

Theorem 2. *Consider a linear system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *with coefficient matrix* $\mathbf{A}$ *of order* $m \times n$*.*

- *When* $r(\mathbf{A}) = n$*, the system is either inconsistent or has exactly one solution.*
- *When* $r(\mathbf{A}) < n$*, the system is either inconsistent or has infinitely many solutions.*

Theorem 3. *Suppose* $\mathbf{A}$ *is the coefficient matrix of a linear system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$*.*
Let $\vec{\mathbf{x}}$ *be a solution of this system and let* $\vec{\mathbf{y}}$ *be another vector. Then* $\vec{\mathbf{y}}$ *is also a solution of the same system if, and only if,* $\vec{\mathbf{x}} - \vec{\mathbf{y}}$ *is in* $N(\mathbf{A})$*.*

In other words, when $\vec{\mathbf{x}}$ *is one solution of the system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$*, then all other solutions must be of the form* $\vec{\mathbf{x}} + \vec{\mathbf{z}}$*, where* $\vec{\mathbf{z}}$ *is in* $N(\mathbf{A})$*.*

Let us illustrate Theorem 1 with an example. Consider a system of the form:

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & b_1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & b_2 \\ & & x_2 & + & 2x_3 & = & b_3 \end{array}$$

Here the coefficient matrix $\mathbf{A}$ and the extended matrix $[\mathbf{A}, \vec{\mathbf{b}}]$ are

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \qquad \text{and} \qquad [\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 2 & 3 & 4 & b_2 \\ 0 & 1 & 2 & b_3 \end{bmatrix}$$

Gaussian elimination on the extended matrix gives:

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 2 & 3 & 4 & b_2 \\ 0 & 1 & 2 & b_3 \end{bmatrix} \overset{R2 \mapsto R2 - 2R1}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 1 & 2 & b_3 \end{bmatrix} \overset{R3 \mapsto R3 - R2}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{bmatrix}$$

Notice that if we delete the last column from this row echelon form of the extended matrix, we obtain the row echelon form of the coefficient matrix $\mathbf{A}$. The first and second columns are pivotal columns, but the third one is not. So we can conclude that $r(\mathbf{A}) = 2$, which is less than the number m of rows of $\mathbf{A}$. Now Theorem 1 indicates that the system will be consistent for some choices of the vector $\vec{\mathbf{b}} = [b_1, b_2, b_3]^T$ and inconsistent for some other choices of this vector. In other words, the column space $CS(\mathbf{A})$ contains some, but not all vectors in $\mathbb{R}^3$.

Question 53.1: What kind of subset of $\mathbb{R}^3$ is the set of vectors $\vec{\mathbf{b}}$ for which the above system is consistent?

Question 53.2: Give an example of a nonzero vector $\vec{\mathbf{b}}$ in $\mathbb{R}^3$ for which the above system is consistent, and an example of a vector $\vec{\mathbf{b}}$ in $\mathbb{R}^3$ for which the above system is inconsistent.

Question 53.3: Suppose $\vec{\mathbf{b}}$ is a vector such that the above system is consistent. What does Theorem 2 then imply about its solution set?