MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 54: THE RANK OF THE COEFFICIENT MATRIX AND THEORY OF SOLUTIONS OF A LINEAR SYSTEM, PART II: HOW TO REPRESENT SOLUTION SETS

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This module is based on Lecture 29. Recall the following theorem from this lecture:

Theorem 1. *Suppose* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is a consistent linear system with a coefficient matrix of order* $m \times n$. *Then the solution set can be described by choosing exactly* $k = dim(N(\mathbf{A})) = n - r(\mathbf{A})$ *among the variables* $x_1, \ldots, x_n$ *as free parameters.*

Question 54.1: Let $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ be a system with extended matrix $[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 0 & b_3 \end{bmatrix}$

(a) Find $r(\mathbf{A})$ and $dim(N(\mathbf{A}))$.
(b) Under what conditions on b_1, b_2, and/or b_3 is the above system consistent?
(c) If the system is consistent, can we describe its solution set by taking:

- x_1 as (one of) the free parameter(s)?
- x_2 as (one of) the free parameter(s)?
- x_3 as (one of) the free parameter(s)?

We also saw in Lecture 29 that we can always pick one particular solution $\vec{\mathbf{x}}$ and a basis $B = \{\vec{\mathbf{z}}_1, \ldots, \vec{\mathbf{z}}_k\}$ of $N(\mathbf{A})$ and then write all solution vectors in the form $\vec{\mathbf{x}} + c_1\vec{\mathbf{z}}_1 + \cdots + c_k\vec{\mathbf{z}}_k$ for some coefficients. The coefficients c_i here will usually be the same as our free variables, but that depends on how the basis for the null space is chosen. Also, the numbering may change; in the second example of Lecture 28 the coefficient c_2 for the second basis vector corresponded to the free variable x_2. To avoid possible confusion, we will always use different letters c_j here for the coefficients and x_i for the free variables.

First we will work out a couple more examples, and then you will be asked to practice the method following the same template.

Consider the following system of linear equations:

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 2x_2 & + & 3x_3 & = & 6 \end{array} \tag{1}$$

We can easily verify that the vector $\vec{\mathbf{x}} = [1, -1, 2]^T$ is a solution. Moreover, the coefficient matrix is the matrix $\mathbf{A}_2$ of Example 2 of Lecture 28, and we already know that the null space $N(\mathbf{A}_2)$ consists of all vectors $\vec{\mathbf{z}}$ of the form $[x_1, -x_1, 0]^T$ and has a basis $\{\vec{\mathbf{z}}_1\} = \{[1, -1, 0]^T\}$.
Thus the null space $N(\mathbf{A}_2)$ will consist of all vectors $\vec{\mathbf{z}}$ of the form $c\vec{\mathbf{z}}_1$, where c is a scalar coefficient.

In view of the above facts, we can represent the solution set of (1) as the set of all vectors of the form

$$\vec{\mathbf{x}} + \vec{\mathbf{z}} = \vec{\mathbf{x}} + c\vec{\mathbf{z}}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+c \\ -1-c \\ 2 \end{bmatrix} \tag{2}$$

We can see that the solution set forms a line in $\mathbb{R}^3$, but not a line through the origin. So the solution set is *not* a linear subspace of $\mathbb{R}^3$. It is an example of what the literature calls an *affine subspace* of $\mathbb{R}^3$.

It is interesting to compare (2) with a representation of the solution set in terms of one free variable that we get from Gaussian elimination on the extended matrix and back-substitution:

$$[\mathbf{A}_2, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 6 \end{bmatrix} \overset{R2 \mapsto R2 - 2R1}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solving the resulting equivalent system

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 2 \\ & & & & x_3 & = & 2 \end{array}$$

by back-substitution and using x_1 as our free variable we conclude that the solution set consists of all vectors of the form

$$\begin{bmatrix} x_1 \\ -x_1 \\ 2 \end{bmatrix} \tag{3}$$

Notice that this representation is *different* from the one in (2). However, if we choose the value x_1 in (3) as $x_1 = 1 + c$, then we get exactly the same expression as in (2).

Now consider the following system of linear equations:

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 7 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & = & 16 \end{array} \tag{4}$$

We can easily verify that the vector $\vec{\mathbf{x}} = [-1, 1, 2, 5]^T$ is a solution. Moreover, the coefficient matrix is the matrix $\mathbf{A}_3$ of Example 3 of Lecture 28, and we already know that the null space $N(\mathbf{A}_3)$ consists of the form $[x_1, -x_1 - x_4, 0, x_4]^T$ and has a basis $\{\vec{\mathbf{z}}_1, \vec{\mathbf{z}}_2\} = \{[1, -1, 0, 0]^T, [0, -1, 0, 1]^T\}$. Thus $N(\mathbf{A}_3)$ consists of all linear combinations $\vec{\mathbf{z}} = c_1\vec{\mathbf{z}}_1 + c_2\vec{\mathbf{z}}_2$.

We can represent the solution set of (4) as the set of all vectors of the form

$$\vec{\mathbf{x}} + \vec{\mathbf{z}} = \vec{\mathbf{x}} + c_1\vec{\mathbf{z}}_1 + c_2\vec{\mathbf{z}}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 5 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 + c_1 \\ 1 - c_1 - c_2 \\ 2 \\ 5 + c_2 \end{bmatrix}$$

We can see that the solution set forms a plane in $\mathbb{R}^4$, but not a plane through the origin. So the solution set is again *not* a linear subspace of $\mathbb{R}^4$. It is another example of an *affine space.*

When attempting the following three problems, you may want to use your answers of Questions 52.2 through 52.7 of Module 52.

Question 54.2: Consider the system

$$\begin{array}{rcrl} x_1 & - & 2x_2 & = -1 \\ 2x_1 & - & 4x_2 & = -2 \end{array} \tag{5}$$

Verify that the vector $\vec{\mathbf{x}} = [1, 1]^T$ is a solution and describe the solution set in terms of linear combinations that use a basis of the null space of the coefficient matrix.

Question 54.3: Consider the system

$$\begin{array}{rcrcrl} x_1 & - & 2x_2 & + & 3x_3 & = -2 \\ 2x_1 & - & 4x_2 & + & x_3 & = 1 \\ 3x_1 & - & 6x_2 & + & 4x_3 & = -1 \end{array} \tag{6}$$

Verify that the vector $\vec{\mathbf{x}} = [1, 0, -1]^T$ is a solution and describe the solution set in terms of linear combinations that use a basis of the null space of the coefficient matrix.

Question 54.4: Consider the system

$$\begin{array}{rcrcrl} x_1 & - & 2x_2 & + & 3x_3 & = -4 \\ 2x_1 & - & 4x_2 & + & 6x_3 & = -8 \\ -x_1 & + & 2x_2 & - & 3x_3 & = 4 \end{array} \tag{7}$$

Verify that the vector $\vec{\mathbf{x}} = [1, 1, -1]^T$ is a solution and describe the solution set in terms of linear combinations that use a basis of the null space of the coefficient matrix.