MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 55A: LINEAR TRANSFORMATIONS

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This module is based on Lecture 30 and Conversation 30A.

Recall that we define in this course a *linear transformation* as a function $L : \mathbb{R}^n \to \mathbb{R}^m$ for some positive integers n, m that satisfies both of the following conditions for all vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}}$ in $\mathbb{R}^n$ and all scalars λ in $\mathbb{R}$:
(i) $L(\lambda\vec{\mathbf{v}}) = \lambda L(\vec{\mathbf{v}})$
(ii) $L(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = L(\vec{\mathbf{v}}) + L(\vec{\mathbf{w}})$.

When $\mathbf{A}$ is an $m \times n$ matrix, then the function $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m$ that is defined by $L(\vec{\mathbf{v}}) = \mathbf{A}\vec{\mathbf{v}}$ is a linear transformation.

Consider the matrices $\quad \mathbf{A} = \begin{bmatrix} -1 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix} \quad$ and $\quad \mathbf{B} = \begin{bmatrix} 0 & -4 \end{bmatrix}$

Question 55.1: From where to where does the linear transformation $L_{\mathbf{A}}$ go?

Question 55.2: From where to where does the linear transformation $L_{\mathbf{B}}$ go?

Now consider the matrices $\quad \mathbf{C} = \begin{bmatrix} 0 & -4 \\ 2 & 0 \end{bmatrix} \quad$ and $\quad \mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Question 55.3: Find a formula for the transformation $L_{\mathbf{C}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

Question 55.4: Give a verbal descriptions of the transformations $L_{\mathbf{C}}$ in terms of what it does to a sheet of material.

Question 55.5: Find a formula for the transformation $L_{\mathbf{D}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

Question 55.6: Give a verbal descriptions of the transformation $L_{\mathbf{D}}$ in terms of what it does to a sheet of material.

Recall the following theorem from Conversation 30A:

Theorem 1 (Matrix representation of linear transformations)**.** *Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. If both the elements of the domain $\mathbb{R}^n$ of L and the function values $L(\vec{\mathbf{x}})$ in $\mathbb{R}^m$ are treated as* column vectors. *Then there exists a matrix $\mathbf{A}$ of order $m \times n$ such that $L = L_{\mathbf{A}}$, that is, $L(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$ for all $\vec{\mathbf{x}}$ in $\mathbb{R}^n$.*

As was mentioned in Conversation 30A, if L is in the assumption, then the columns of the matrix $\mathbf{A}$ for which $L = L_{\mathbf{A}}$ are the function values $L(\vec{\mathbf{e}}_i)$ of the standard basis vectors.

Question 55.7: Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$ and $L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix}$ Find the matrix $\mathbf{A}$ such that $L = L_{\mathbf{A}}$.

Question 55.8: Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $L\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$ and $L\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix}$ Find the matrix $\mathbf{A}$ such that $L = L_{\mathbf{A}}$. *Hint:* Here you need to find first express $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2$ as linear combinations of the input vectors for which the function values are given and then find the values $L(\vec{\mathbf{e}}_1)$ and $L(\vec{\mathbf{e}}_2)$ by using the assumption that L is a linear transformation.

Question 55.9: Suppose $L : \mathbb{R}^3 \to \mathbb{R}^2$ is such that $L(\vec{\mathbf{e}}_1) = \begin{bmatrix}2\\3\end{bmatrix}$ $L(\vec{\mathbf{e}}_2) = L(\vec{\mathbf{e}}_3) = -2L(\vec{\mathbf{e}}_1)$. Find the matrix $\mathbf{A}$ such that $L = L_{\mathbf{A}}$.