MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 55B: MATRIX REPRESENTATION OF LINEAR TRANSFORMATIONS

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This module is based on Lecture 30 and Conversation 30A. Recall the following theorem from this Conversation:

Theorem 1 (Matrix representation of linear transformations)**.** *Suppose* $L : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear transformation. If both the elements of the domain* $\mathbb{R}^n$ *of* L *and the function values* $L(\vec{\mathbf{x}})$ *in* $\mathbb{R}^m$ *are treated as* column vectors. *Then there exists a matrix* $\mathbf{A}$ *of order* $m \times n$ *such that* $L = L_{\mathbf{A}}$, *that is,* $L(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$ *for all* $\vec{\mathbf{x}}$ *in* $\mathbb{R}^n$.

Proof of Theorem 1: Let us analyze a proof of this theorem here.

Recall that if we treat $\mathbb{R}^n$ as a space of column vectors, then the standard basis vectors are

$$\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad \vec{\mathbf{e}}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Thus every vector $\vec{\mathbf{x}} \in \mathbb{R}^n$ can be uniquely expressed as a linear combination

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1\vec{\mathbf{e}}_1 + x_2\vec{\mathbf{e}}_2 + \dots + x_n\vec{\mathbf{e}}_n.$$

Now let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Consider the $m \times 1$ column vectors

$$L(\vec{\mathbf{e}}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \quad L(\vec{\mathbf{e}}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \quad \dots \quad L(\vec{\mathbf{e}}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Let us form an $m \times n$ matrix $\mathbf{A}$ with these columns:

$$\mathbf{A} = [L(\vec{\mathbf{e}}_1), L(\vec{\mathbf{e}}_2), \dots, L(\vec{\mathbf{e}}_n)] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

We want to show that L is the same linear transformation as the linear transformation $L_{\mathbf{A}}$ that is defined by $L_{\mathbf{A}}(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$ for every vector $\vec{\mathbf{x}} \in \mathbb{R}^n$.

Let us first calculate the values $L_{\mathbf{A}}(\vec{\mathbf{e}}_j)$ for the standard basis vectors $\vec{\mathbf{e}}_j$.

$$(1) \qquad L_{\mathbf{A}}(\vec{\mathbf{e}}_1) = \mathbf{A}\vec{\mathbf{e}}_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = L(\vec{\mathbf{e}}_1)$$

Question 55.10: Where does the last equality in (1) come from?

Similarly,

$$L_{\mathbf{A}}(\vec{\mathbf{e}}_2) = \mathbf{A}\vec{\mathbf{e}}_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} = L(\vec{\mathbf{e}}_2)$$

(and so on ...)

$$L_{\mathbf{A}}(\vec{\mathbf{e}}_n) = \mathbf{A}\vec{\mathbf{e}}_n = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = L(\vec{\mathbf{e}}_n)$$

We have shown that for $j = 1, 2, \dots, n$:

$$L_{\mathbf{A}}(\vec{\mathbf{e}}_j) = \mathbf{A}\vec{\mathbf{e}}_j = L(\vec{\mathbf{e}}_j).$$

Now let $\vec{\mathbf{x}}$ be any vector in $\mathbb{R}^n$. Then

$$\vec{\mathbf{x}} = x_1\vec{\mathbf{e}}_1 + x_2\vec{\mathbf{e}}_2 + \cdots + x_n\vec{\mathbf{e}}_n,$$

and it follows from the linearity of $L_{\mathbf{A}}$ that

$$L_{\mathbf{A}}(\vec{\mathbf{x}}) = x_1 L_{\mathbf{A}}(\vec{\mathbf{e}}_1) + x_2 L_{\mathbf{A}}(\vec{\mathbf{e}}_2) + \cdots + x_n L_{\mathbf{A}}(\vec{\mathbf{e}}_n).$$

From what we have already shown it follows that

$$(2) \qquad L_{\mathbf{A}}(\vec{\mathbf{x}}) = x_1 L(\vec{\mathbf{e}}_1) + x_2 L(\vec{\mathbf{e}}_2) + \cdots + x_n L(\vec{\mathbf{e}}_n) = L(\vec{\mathbf{x}}).$$

Question 55.11: Where does the last equality in (2) come from?

Thus the two functions $L_{\mathbf{A}}$ and L take the same values for all inputs $\vec{\mathbf{x}}$ and must be the same function. $\square$