MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 6: ADJACENCY MATRICES AND DEGREE SEQUENCES OF DIRECTED GRAPHS

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We will use here the notation and terminology of Lecture 3, Module 4, and Conversation 3.

Consider the directed graph aka digraph that was discussed in Conversation 3:

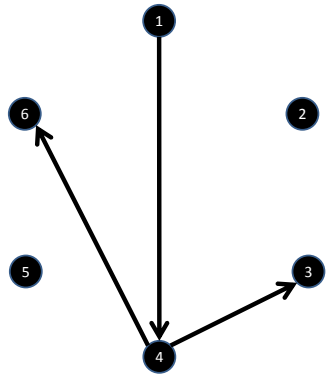

FIGURE 1. The directed graph of Conversation 3.

When you look at this digraph, you will see that the notion of "degree" of a node becomes ambiguous. Node 4 is the source of two arcs and the target of one arc, and neither defining the degree as 2 nor defining it as 1 would give us a complete picture.

Therefore, we will say that node 4 has *outdegree* 2 and *indegree* 1. Symbolically: $outdeg(4) = 2$ and $indeg(4) = 1$. Formally, $outdeg(i)$ is defined as the number of arcs with source i and $indeg(i)$ is defined as the number of arcs with target i.

Question 6.1: Find $outdeg(i)$ and $indeg(i)$ for the remaining nodes of this digraph.

Recall that for constructing the adjacency matrix $\mathbf{A} = [a_{ij}]$ of a digraph we let a_{ij} be the number of arcs arcs with source i and target j. For the digraph above we obtained:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [a_{ij}]_{6\times 6}$$

Question 6.2: The matrix product $\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}\mathbf{A}$ is a row vector. What does it represent?

Question 6.3: The matrix product $\mathbf{A}\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}$ is a column vector. What does it represent?

Consider the following matrix: $\mathbf{A} = \begin{bmatrix}0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$

Either $\mathbf{A}$ or $\mathbf{A}^T$ is the adjacency matrix of a digraph whose nodes represent a group of 4 females in such a way that an arc from i to j indicates that i is the mother of j.

Question 6.4: Based on this information, how can you tell whether $\mathbf{A}$ or $\mathbf{A}^T$ is the correct adjacency matrix? *Hint:* Start by sketching digraphs with adjacency matrices $\mathbf{A}$ and $\mathbf{A}^T$.

So far, we have seen only examples of simple digraphs with at most one arc from a given source i to a given target j. Now let us consider a different kind of digraph.

Go to the the mock-up of a class website `https://people.ohio.edu/just/M3200S18Mock/`.

This website has five pages that we will represent by nodes that are numbered as follows:
1 = class home, 2 = syllabus, 3 = homework, 4 = supplements, 5 = sample solutions.
We can then construct a digraph by drawing a *separate* arc from node i to node j for each link on page i that points to page j. In this digraph there will be multiple arcs with the same source and target. For example, notice that there are many links from the homework page (node 3) to the supplements (node 4).

Question 6.5: Find the adjacency matrix $\mathbf{A}$ of this directed graph.

Question 6.6: What is the meaning of the outdegree $outdeg(i)$ of node i in this example?

Question 6.7: What is the meaning of the indegree $indeg(i)$ of node i in this example?

Question 6.8: How could you calculate the sequence of outdegrees as a matrix product here?

Question 6.9: How could you calculate the sequence of indegrees as a matrix product here?