MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 62: COMPUTING DETERMINANTS BY PIVOTAL CONDENSATION

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This module is based on Lecture 32. Recall from this lecture that the determinant can be defined as a function that assigns to each square matrix a number $\det(\mathbf{A})$.

When $\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ we also use the notation $\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$ for $\det\left(\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}\right)$

Note that it makes a big difference whether we enclose the elements of $\mathbf{A}$ in square brackets or in vertical lines; in the former case we get a matrix, in the latter we get a number.

The function det behaves under elementary row operations in the following way:

(a) If $\mathbf{E}$ implements elementary row operation (E1): "Exchange two rows of $\mathbf{A}$," then $\det(\mathbf{EA}) = -\det(\mathbf{A})$.
(b) If $\mathbf{E}$ implements elementary row operation (E2): "Multiply one row of $\mathbf{A}$ by a nonzero scalar $\lambda \neq 0$," then $\det(\mathbf{EA}) = \lambda \det(\mathbf{A})$
(c) If $\mathbf{E}$ implements elementary row operation (E3): "Add a scalar multiple of one row of $\mathbf{A}$ to another row of $\mathbf{A}$," then $\det(\mathbf{EA}) = \det(\mathbf{A})$.
(d) If $\mathbf{A}$ is upper-triangular or lower-triangular, then $\det(\mathbf{A})$ is the product of the diagonal elements.

These properties allow us to calculate $\det(\mathbf{A})$ for every square matrix $\mathbf{A}$ with a step-by-step procedure called *pivotal condensation* that works as follows:

(1) Transform $\mathbf{A}$ by successive elementary row operations into
$$\mathbf{A} \to \mathbf{E}_1\mathbf{A} \to \mathbf{E}_2\mathbf{E}_1\mathbf{A} \to \cdots \to \mathbf{E}_k \dots \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U}, \text{ where } \mathbf{U} \text{ is upper-triangular.}$$
(2) Use Properties (a)–(c) above to keep track of how the determinant changes at every step.
(3) Calculate $\det(\mathbf{U})$ as the product of the diagonal elements.
(4) Deduce $\det(\mathbf{A})$ from $\det(\mathbf{U})$ and your record on how the determinant did or did not change at every step.

A worked-out example was given in Lecture 32. Here we will practice this method.

Question 62.1: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ Use pivotal condensation to find $\det(\mathbf{A})$.

Question 62.2: Suppose $\mathbf{A}$ is a 3×3 matrix, and that after dividing Row 1 by 2, switching Rows 2 and 3, adding 5 times Row 1 to Row 3, and then multiplying Row 3 by 7, we obtain

the matrix $\mathbf{U} = \begin{bmatrix} 7 & 2 & -3 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ Find $\det(\mathbf{A})$.

Question 62.3: Let $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Use pivotal condensation to find $\det(\mathbf{B})$.

Question 62.4: Let $\mathbf{C} = \begin{bmatrix} 0 & -3 & -8 \\ 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ Use pivotal condensation to find $\det(\mathbf{C})$.

Question 62.5: Let $\mathbf{D} = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 55 & 55 & 55 & 55 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 0 & -3 \end{bmatrix}$ Use pivotal condensation to find $\det(\mathbf{D})$.

Question 62.6: Consider the argument given below. Is it correct? If not, which step contains a mistake?

Let $\mathbf{A} = \begin{bmatrix} 0 & -5 & -4 \\ 11 & 22 & 33 \\ -1 & -7 & 6 \end{bmatrix}$ We use pivotal condensation to find $\det(\mathbf{A})$.

Step 1: Switch rows 1 and 2. As a result of this operation, the sign of the determinant changes and we get:

$$\det(\mathbf{A}) = \begin{vmatrix} 0 & -5 & -4 \\ 11 & 22 & 33 \\ -1 & -7 & 6 \end{vmatrix} = - \begin{vmatrix} 11 & 22 & 33 \\ 0 & -5 & -4 \\ -1 & -7 & 6 \end{vmatrix}$$

Step 2: Divide row 1 by 11. As a result of this operation, the determinant changes by a factor of $\frac{1}{11}$ and we get:

$$\det(\mathbf{A}) = \begin{vmatrix} 0 & -5 & -4 \\ 11 & 22 & 33 \\ -1 & -7 & 6 \end{vmatrix} = - \begin{vmatrix} 11 & 22 & 33 \\ 0 & -5 & -4 \\ -1 & -7 & 6 \end{vmatrix} = -\frac{1}{11} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -4 \\ -1 & -7 & 6 \end{vmatrix}$$

Step 3: Add row 1 to row 3. As a result of this operation, the determinant does not change and we get:

$$\det(\mathbf{A}) = \begin{vmatrix} 0 & -5 & -4 \\ 11 & 22 & 33 \\ -1 & -7 & 6 \end{vmatrix} = -\frac{1}{11} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -4 \\ -1 & -7 & 6 \end{vmatrix} = -\frac{1}{11} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -4 \\ 0 & -5 & 9 \end{vmatrix}$$

Step 4: Subtract row 2 from row 3. As a result of this operation, the determinant does not change and we get:

$$\det(\mathbf{A}) = \begin{vmatrix} 0 & -5 & -4 \\ 11 & 22 & 33 \\ -1 & -7 & 6 \end{vmatrix} = -\frac{1}{11} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -4 \\ 0 & -5 & 9 \end{vmatrix} = -\frac{1}{11} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 13 \end{vmatrix}$$

Step 5: In previous step we obtained an upper-triangular matrix. Its determinant is the product of its diagonal elements, and we get:

$$\det(\mathbf{A}) = \begin{vmatrix} 0 & -5 & -4 \\ 11 & 22 & 33 \\ -1 & -7 & 6 \end{vmatrix} = -\frac{1}{11}\begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 13 \end{vmatrix} = -\frac{1}{11}(1)(-5)(13) = \frac{65}{11}$$