MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 63A: MORE PROPERTIES OF DETERMINANTS

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This module is based on Lecture 33. Recall the following properties of determinants from this lecture:

Theorem 1. *Let $\mathbf{A}$ be any square matrix. Then:*

(1) If $\mathbf{B}$ is obtained by exchanging two rows of $\mathbf{A}$ or two columns of $\mathbf{A}$, then $\det(\mathbf{B}) = -\det(\mathbf{A})$.
(2) If $\mathbf{B}$ is obtained by multiplying one row of $\mathbf{A}$ or one column of $\mathbf{A}$ by a scalar λ, then $\det(\mathbf{B}) = \lambda \det(\mathbf{A})$.
(3) If $\mathbf{B}$ is obtained by adding a scalar multiple of one row of $\mathbf{A}$ to another row of $\mathbf{A}$ or one column of $\mathbf{A}$ to another column of $\mathbf{A}$, then $\det(\mathbf{B}) = \det(\mathbf{A})$.

Theorem 2. *Let $\mathbf{A}, \mathbf{B}$ be matrices of order $n \times n$. Then*

(1) $\det(\mathbf{A}^T) = \det(\mathbf{A})$.
(2) $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.
(3) If $\mathbf{A}^{-1}$ exists, then $\det(\mathbf{A}^{-1}) = \frac{1}{\det(A)}$.

Theorem 3. *The following properties of an $n \times n$ matrix $\mathbf{A}$ are equivalent:*

(1) $\det(\mathbf{A}) = 0$; that is, $\mathbf{A}$ is singular.
(2) $r(\mathbf{A}) < n$.
(3) $\mathbf{A}$ is not invertible, that is, $\mathbf{A}^{-1}$ does not exist.
(4) The system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is underdetermined.
(5) Each system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is either underdetermined or inconsistent.
(6) The range of $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ is not all of $\mathbb{R}^n$.
(7) The function $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ is not one-to-one.

Theorem 4. *The following properties of an $n \times n$ matrix $\mathbf{A}$ are equivalent:*

(1) $\mathbf{A}$ is non-singular, *that is,* $\det(\mathbf{A}) \neq 0$.
(2) $\mathbf{A}$ has full rank, that is, $r(\mathbf{A}) = n$.
(3) $\mathbf{A}$ is invertible, that is, $\mathbf{A}^{-1}$ exists.
(4) The system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a unique solution.
(5) Each system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution.
(6) The range of $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbb{R}^n$.
(7) The function $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one.

Question 63.1: Which of the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ of Questions 62.1–62.4 of Module 62 is/are invertible?

For Questions 63.2–63.7, consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{bmatrix}$

It can be shown that $\det(\mathbf{A}) = -78$.

Question 63.2: Based on this information, find $\begin{vmatrix} 2 & 5 & 3 \\ 7 & 13 & 11 \\ 17 & 23 & 19 \end{vmatrix}$

Question 63.3: Based on this information, find $\begin{vmatrix} 1 & 7 & 17 \\ 1.5 & 11 & 19 \\ 2.5 & 13 & 23 \end{vmatrix}$

Question 63.4: Based on this information, find $\begin{vmatrix} 7 & 5 & 3 \\ 20 & 13 & 11 \\ 40 & 23 & 19 \end{vmatrix}$

Question 63.5: Based on this information, what can you say about $\mathbf{A}^{-1}$ and its determinant?

Question 63.6: Based on this information, determine whether the following system of linear equations is underdetermined, inconsistent, or has exactly one solution:

$$\begin{array}{rcrcrcl} 2x_1 & + & 3x_2 & + & 5x_3 & = & 1 \\ 7x_1 & + & 11x_2 & + & 13x_3 & = & 4 \\ 17x_1 & + & 19x_2 & + & 23x_3 & = & 8 \end{array}$$

Question 63.7: (a) Can you conclude, based on the above information, that there exists at least one vector $\vec{\mathbf{x}} \in \mathbb{R}^3$ such that $L_{\mathbf{A}}(\vec{\mathbf{x}}) = \begin{bmatrix} \pi \\ 0.6 \\ 45 \end{bmatrix}$?

(b) Can you conclude, based on the above information, that there exists at most one vector $\vec{\mathbf{x}} \in \mathbb{R}^3$ such that $L_{\mathbf{A}}(\vec{\mathbf{x}}) = \begin{bmatrix} \pi \\ 0.6 \\ 45 \end{bmatrix}$?

An *LU-decomposition* of an $n \times n$ matrix $\mathbf{A}$ is a pair of $n \times n$ matrices $\mathbf{L}$ and $\mathbf{U}$ such that: $\mathbf{L}$ is lower-triangular and $\mathbf{U}$ is upper-triangular and $\mathbf{A} = \mathbf{LU}$. If $\mathbf{A}$ has an LU-decomposition, then $\mathbf{L}$ and $\mathbf{U}$ can be chosen in such a way that all elements on the (main) diagonal of $\mathbf{L}$ are equal to 1.

Question 63.8: Suppose $\mathbf{A}$ is an $n \times n$ matrix with an LU decomposition such that *both* $\mathbf{L}$ and $\mathbf{U}$ have only ones on their (main) diagonals. Based on this information, what can one deduce about $\det(\mathbf{A})$?