MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 64: DETERMINANTS AND PROPERTIES OF LINEAR TRANSFORMATIONS

WINFRIED JUST, OHIO UNIVERSITY

This module is based on Lecture 34. Recall the following theorems from this lecture:

Theorem 1. *Let* $\mathbf{A}$ *be a matrix of order* 2×2 *and let* $L_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^2$ *be the transformation of the Euclidean plane that is defined by* $\mathbf{A}$. *Then*

(i) $L_{\mathbf{A}}$ *maps any region of area* A *onto a region of area* $|\det(\mathbf{A})|A$.

(ii) If $\det(\mathbf{A}) > 0$, *then* $L_{\mathbf{A}}$ preserves *orientation.*

(iii) If $\det(\mathbf{A}) < 0$, *then* $L_{\mathbf{A}}$ reverses *orientation.*

(iv) If $\det(\mathbf{A}) = 0$, *then* $L_{\mathbf{A}}$ *maps* $\mathbb{R}^2$ *to a lower-dimensional subspace.*

Theorem 2. *Let* $\mathbf{A}$ *be a matrix of order* 3×3 *and let* $L_{\mathbf{A}} : \mathbb{R}^3 \to \mathbb{R}^3$ *be the transformation of the Euclidean space that is defined by* $\mathbf{A}$. *Then*

(i) $L_{\mathbf{A}}$ *maps any region of volume* V *onto a region of volume* $|\det(\mathbf{A})|V$.

(ii) If $\det(\mathbf{A}) > 0$, *then* $L_{\mathbf{A}}$ preserves *orientation.*

(iii) If $\det(\mathbf{A}) < 0$, *then* $L_{\mathbf{A}}$ reverses *orientation.*

(iv) If $\det(\mathbf{A}) = 0$, *then* $L_{\mathbf{A}}$ *maps* $\mathbb{R}^3$ *to a lower-dimensional subspace.*

Here the transformation could be implemented as a continuous deformation and/or rotation of solid objects if, and only if, it preserves orientation.

The matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ defines a linear transformation $L_{\mathbf{A}}$ that maps the rectangle R with vertices $[0,0], [0,3], [4,0], [4,3]$ onto a quadrilateral Q.

Question 64.1: Find the area of Q.

Question 64.2: Does the transformation $L_{\mathbf{A}}$ preserve or reverse orientation?

The matrix $\mathbf{B} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ defines a linear transformation $L_{\mathbf{B}}$ that maps the disc D whose center is the origin and whose radius is $r = 2$ onto a region E whose boundary is an ellipse.

Question 64.3: Find the area of the region E.

Question 63.4: Does the map $L_{\mathbf{B}}$ preserve or reverse orientation?

Question 64.5: Consider the diagonal matrix $\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

How does $L_{\mathbf{D}}$ change volumes, if at all? Does $L_{\mathbf{D}}$ preserve orientation? Does $L_{\mathbf{D}}$ change orientation?

Question 64.6: Consider the matrix $\mathbf{B}$ of Question 62.3 of Module 62. How does $L_{\mathbf{B}}$ change volumes, if at all? Does $L_{\mathbf{B}}$ preserve orientation? Does $L_{\mathbf{B}}$ change orientation?

Question 64.7: Consider the matrix $\mathbf{C}$ of Question 62.4 of Module 62. How does $L_{\mathbf{C}}$ change volumes, if at all? Does $L_{\mathbf{C}}$ preserve orientation? Does $L_{\mathbf{C}}$ change orientation?