MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 65: CALCULATING DETERMINANTS BY COFACTOR EXPANSION

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This module is based on Lecture 35.

Recall that a *minor* of a matrix $\mathbf{A}$ is a determinant of any square submatrix of $\mathbf{A}$.

The *cofactor of the element* a_{ij} of a square matrix $\mathbf{A}$ is the product of $(-1)^{i+j}$ with the minor that is obtained by removing the i^{th} row and the j^{th} column of $\mathbf{A}$.

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 2 & 6 \\ 5 & 3 & 1 \end{bmatrix}$

Question 65.1: For the above matrix $\mathbf{A}$, find the cofactor of the element a_{12}.

Question 65.2: For the above matrix $\mathbf{A}$, find the cofactor of the element a_{31}.

For any square matrix $\mathbf{A}$ one can calculate $\det(\mathbf{A})$ by *cofactor expansion* as follows:

(1) Pick any row or column.
(2) For each element of the chosen row or column, find its cofactor.
(3) Multiply each element in the chosen row or column by its cofactor.
(4) Sum the results.

The method works best if you choose the row or column along which you expand as one that contains many zero elements.

Question 65.3: Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ -3 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ Compute $\det(\mathbf{A})$ by cofactor expansion.

Question 65.4: Let $\mathbf{A} = \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 7 & 0 & 3 \\ 4 & 0 & 0 & 5 \\ -3 & 2 & 3 & 0 \end{bmatrix}$ Compute $\det(\mathbf{A})$ by cofactor expansion.

Question 65.5: Consider the following calculation by cofactor expansion. Is it correct?
If not, which step(s) contain(s) (a) mistake(s)?

Let $\mathbf{A} = \begin{bmatrix} -1 & 2 & 1 & 4 & 0 \\ 0 & 7 & 0 & 3 & 0 \\ 4 & 0 & 0 & 5 & -4 \\ -3 & 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$ We compute $\det(\mathbf{A})$ by cofactor expansion.

Step 1: To find $\det(\mathbf{A}) = \begin{vmatrix} -1 & 2 & 1 & 4 & 0 \\ 0 & 7 & 0 & 3 & 0 \\ 4 & 0 & 0 & 5 & -4 \\ -3 & 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$ expand along the fifth column:

$$\det(\mathbf{A}) = 0 + 0 + -4(-1)^{3+5} \begin{vmatrix} -1 & 2 & 1 & 4 \\ 0 & 7 & 0 & 3 \\ -3 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} + 0 + 0.$$

Step 2: Now compute the relevant minor (determinant of the 4×4 submatrix) by expanding along the second row:

$$\begin{vmatrix} -1 & 2 & 1 & 4 \\ 0 & 7 & 0 & 3 \\ -3 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0 + 7(-1)^{2+2} \begin{vmatrix} -1 & 1 & 4 \\ -3 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} + 0 + 3(-1)^{2+4} \begin{vmatrix} -1 & 2 & 1 \\ -3 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

Steps 3 and 4: Now compute the new relevant minors (determinants of the 3×3 submatrices) again by cofactor expansion:

Step 3: For the first one let's expand along the third column:

$$\begin{vmatrix} -1 & 1 & 4 \\ -3 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 4(-1)^{1+3} \begin{vmatrix} -3 & 3 \\ 1 & 1 \end{vmatrix} + 0 + 1(-1)^{3+3} \begin{vmatrix} -1 & 1 \\ -3 & 3 \end{vmatrix} = 4(1)(-6) + 1(1)(0) = -24.$$

Step 4: For the second one, let's expand along the third row:

$$\begin{vmatrix} -1 & 2 & 1 \\ -3 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 1(-1)^{3+1} \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 1(-1)^{3+2} \begin{vmatrix} -1 & 1 \\ -3 & 3 \end{vmatrix} + 1(-1)^{3+3} \begin{vmatrix} -1 & 2 \\ -3 & 2 \end{vmatrix} = 4 - 0 + 4 = 8.$$

Step 5: Now substitute these results in the formula for the first minor:

$$\begin{vmatrix} -1 & 2 & 1 & 4 \\ 0 & 7 & 0 & 3 \\ -3 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 7 \begin{vmatrix} -1 & 1 & 4 \\ -3 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 2 & 1 \\ -3 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 7(-24) + 3(8) = -144.$$

Step 6: Finally, substitute these results into the formula for $\det(\mathbf{A})$:

$$\det(\mathbf{A}) = -4 \begin{vmatrix} -1 & 2 & 1 & 4 \\ 0 & 7 & 0 & 3 \\ -3 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -4(-144) = -576.$$