MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 66: INTRODUCTION TO EIGENVECTORS AND EIGENVALUES

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This module is based on Lecture 36 and Conversation 31. Recall the following definition from Lecture 36:

Definition 1. *A vector* $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ *is an* eigenvector *of a square matrix* $\mathbf{A}$ *if there exists a scalar* λ *such that* $\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$. *Then* λ *is an* eigenvalue *of* $\mathbf{A}$.

Note that an eigenvalue is allowed to be 0, but eigenvectors are not allowed to be zero vectors. Checking whether a given vector $\vec{\mathbf{x}}$ is an eigenvector of a given matrix $\mathbf{A}$ and finding its eigenvalue is straightforward.

Question 66.1: Let $\mathbf{A} = \begin{bmatrix} -0.25 & -3.50 & -3.75 \\ 0.25 & 3.50 & 3.75 \\ -0.25 & 3.50 & 3.25 \end{bmatrix}$

Which of the vectors $\vec{\mathbf{x}}_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ $\vec{\mathbf{x}}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ $\vec{\mathbf{x}}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ $\vec{\mathbf{x}}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

are eigenvectors of $\mathbf{A}$, and what are their eigenvalues?

Question 66.2: Let $\mathbf{A}$ be the matrix of Question 66.1. Based on the answer to Question 66.1 and what you learned in Conversation 31, verbally describe what the corresponding linear transformation $L_{\mathbf{A}} : \mathbb{R}^3 \to \mathbb{R}^3$ does.

Recall the following facts from Lecture 36:

Proposition 1. *Let* $\mathbf{A}$ *be a square matrix. If* $\vec{\mathbf{x}}$ *is an eigenvector of* $\mathbf{A}$ *with eigenvalue* λ, *then for every scalar* $c \neq 0$ *the vector* $c\vec{\mathbf{x}}$ *is also an eigenvector of* $\mathbf{A}$ *with the same eigenvalue* λ.

Proposition 2. *Let* $\mathbf{A}$ *be a square matrix. The matrix* $\mathbf{A}$ *has an eigenvalue* $\lambda = 0$ *if, and only if,* $\mathbf{A}$ *is singular.*

Definition 2. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. We say that* $\mathbf{A}$ *has a* full set of eigenvectors *if there exist* n *eigenvectors of* $\mathbf{A}$ *that form a linearly independent set.*

Theorem 3. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix (whose elements are reals). Assume* $\mathbf{A}$ *has* n *pairwise distinct real eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$. *Then* $\mathbf{A}$ *has a full set of eigenvectors.*

Question 66.3: Suppose $\vec{\mathbf{x}} = \begin{bmatrix} 13 \\ -39 \\ 26 \end{bmatrix}$ is an eigenvector of a given matrix $\mathbf{A}$ with eigenvalue $\lambda = 4$. Find an eigenvector $\vec{\mathbf{y}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ of $\mathbf{A}$ with eigenvalue $\lambda = 4$ such that $y_1 = 1$.

Question 66.4: Let $\mathbf{A} = \begin{bmatrix} 1 & 8 & 2 \\ 0 & 0 & 14 \\ 0 & 0 & -3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 20 & 30 & 0 \\ 66 & 0 & -33 \end{bmatrix}$

Which of the above matrices has/have eigenvectors with eigenvalue 0?

Proposition 1 has a natural generalization:

Proposition 4. *Let* $\mathbf{A}$ *be a square matrix and let* $\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_k$ *be eigenvectors of* $\mathbf{A}$ *that all have the same eigenvalue* λ. *Then every nonzero vector* $\vec{\mathbf{y}}$ *in span*$(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_n)$ *is an eigenvector of* $\mathbf{A}$ *with eigenvalue* λ.

Question 66.5: Prove Proposition 4. *Hint:* You may want to first construct the proof for the special case $k = 2$. In your proof, you will need to use the definition of eigenvectors and eigenvalues and properties of matrix multiplication.

Finally, let us prove Theorem 3 for the special case when $n = 2$.

Question 66.6: (Challenge) Let $\mathbf{A}$ be a 2×2 matrix with eigenvalues $\lambda_1 \neq \lambda_2$, and let $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2$ be eigenvectors with eigenvalues λ_1, λ_2, respectively. Prove that the set $\{\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2\}$ is linearly independent. *Hint:* Think about what it would take for the set $\{\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2\}$ *not* to be linearly independent, and then show that this is impossible.