MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 67A: FINDING EIGENVALUES

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This module is based on Lectures 36 and 37A. Recall from this lecture that the eigenvalues of a square matrix $\mathbf{A}$ are the roots of its *characteristic polynomial* $\det(\mathbf{A} - \lambda\mathbf{I})$ and can be obtained by factoring it.

Here $\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$ is obtained by subtracting the variable λ

from each diagonal element a_{ii} of $\mathbf{A}$.

In this module we will first briefly review some basic facts from pre-calculus about factoring polynomials, then we will practice finding eigenvalues, and finally we will explore some interesting connections between the eigenvalues, the trace, and the determinant of a square matrix.

1. Self-Study: Review of Roots of Polynomials

Recall that the roots λ_1, λ_2 of a quadratic polynomial $a\lambda^2 + b\lambda + c$ are given by

$$\lambda_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \lambda_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

When $b^2 > ac$ we obtain two distinct real numbers $\lambda_1 \neq \lambda_2$, when $b^2 < ac$ we obtain a pair of *conjugate complex numbers* $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$, and when $b^2 = ac$, we obtain only one real root $\lambda = \lambda_1 = \lambda_2$ *of multiplicity 2.*

In general, any polynomial of degree n with real coefficients $a_0, a_1, \dots, a_n$ can be factored as

$a_n\lambda^n + \dots + a_1\lambda + a_0 = a_n(\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$, where each root λ_i is a complex number.

When $\lambda_i = \alpha + \beta i$ is not real, that is, when the imaginary part $\beta \neq 0$, then the *conjugate* $\alpha - \beta i$ also appears as a root λ_j in the above factorization. The number k of times a given root appears in the above factorization is called its *multiplicity.*

For example, $\lambda^5 + 3\lambda^4 + 3\lambda^3 + \lambda^2 = (\lambda - 0)^2(\lambda + 1)^3 = (\lambda - 0)(\lambda - 0)(\lambda + 1)(\lambda + 1)(\lambda + 1)$.

Here $\lambda_1 = \lambda_2 = 0$ is a root of multiplicity $k = 2$ and $\lambda_3 = \lambda_4 = \lambda_5 = -1$ is a root of multiplicity $k = 3$.

When a root λ of the characteristic polynomial of a matrix $\mathbf{A}$ has multiplicity $k > 1$, then would also say that $\lambda = \lambda_1 = \lambda_2$ is a *repeated eigenvalue* or an *eigenvalue of multiplicity* k of $\mathbf{A}$.

2. Practice: Finding Eigenvalues

Now let us practice finding the set of eigenvalues for some specific matrices. This involves finding formulas for determinants of matrices that have elements on the diagonal with a variable λ and makes pivotal condensation difficult. But you still can use cofactor expansion and the fact that the determinant of an upper- or lower-triangular matrix is the product of the elements on the (main) diagonal.

Question 67.1: Find the characteristic polynomial and eigenvalues of $\mathbf{A} = \begin{bmatrix} 8 & 3 \\ -6 & -1 \end{bmatrix}$

Question 67.2: Find the characteristic polynomial and eigenvalues of $\mathbf{B} = \begin{bmatrix} -10 & 10 \\ 0.1 & -0.1 \end{bmatrix}$

Question 67.3: Find the characteristic polynomial and eigenvalues of $\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

Question 67.4: Find the characteristic polynomial and eigenvalues of $\mathbf{D} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 5 & 0 & 0 \\ 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$