MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 67B: FINDING EIGENVECTORS

WINFRIED JUST, OHIO UNIVERSITY

This module is based on Lecture 67B. In this lecture we learned the following step-by-step procedure for finding eigenvectors of a given square matrix $\mathbf{A}$:

(1) Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I})$.
(2) Factor the characteristic polynomial. The roots are the eigenvalues of $\mathbf{A}$.
(3) For each (real) eigenvalue λ_i, find the eigenvectors $\vec{\mathbf{x}}$ with eigenvalue λ_i as follows:
   (a) Form $\mathbf{A} - \lambda_i\mathbf{I}$ by subtracting the number λ_i from each diagonal element of $\mathbf{A}$.
   (b) Solve the system of linear equations $(\mathbf{A} - \lambda_i\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$, for example by Gaussian elimination.
   (c) Your solution will contain *at least 1* and *up to* k_i variables x_j that you can choose arbitrarily. Here k_i denotes the multiplicity of eigenvalue λ_i. For each of these variables x_j, find an eigenvector by setting it to 1, while setting the other variables that you can choose freely to 0.

We practiced the first two steps in the previous section; here we will focus on step (3). Let us first illustrate this step with an example.

Let $\mathbf{C} = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 7 & 8 \\ 4 & 0 & 3 \end{bmatrix}$ In Lecture 37A we found the eigenvalues $\lambda_1 = 7, \lambda_2 = 5, \lambda_3 = -1$.

In Lecture 37B we found that every vector of the form $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$ with $x_2 \neq 0$ is an eigenvector of $\mathbf{C}$ with eigenvalue $\lambda_1 = 7$. In particular, $\vec{\mathbf{x}}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 7$.

Now let us find an eigenvector with eigenvalue $\lambda_2 = 5$:

$$\text{Form } \mathbf{C} - \lambda_2\mathbf{I} = \mathbf{C} - 5\mathbf{I} = \begin{bmatrix} 1-5 & 0 & 2 \\ 6 & 7-5 & 8 \\ 4 & 0 & 3-5 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 2 \\ 6 & 2 & 8 \\ 4 & 0 & -2 \end{bmatrix}$$

The system $(\mathbf{C} - 5\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$ can be written as

$$\begin{array}{rcrcrcl} -4x_1 & & + & & 2x_3 & = & 0 \\ 6x_1 & + & 2x_2 & + & 8x_3 & = & 0 \\ 4x_1 & & - & & 2x_3 & = & 0 \end{array}$$

Perform Gaussian elimination on the extended matrix of this system:

$$\begin{bmatrix} -4 & 0 & 2 & 0 \\ 6 & 2 & 8 & 0 \\ 4 & 0 & -2 & 0 \end{bmatrix} \overset{R2 \mapsto R2+1.5R1}{\longrightarrow} \begin{bmatrix} -4 & 0 & 2 & 0 \\ 0 & 2 & 11 & 0 \\ 4 & 0 & -2 & 0 \end{bmatrix} \overset{R3 \mapsto R3+R1}{\longrightarrow} \begin{bmatrix} -4 & 0 & 2 & 0 \\ 0 & 2 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1\mapsto R1/(-4)} \begin{bmatrix} 1 & 0 & -0.5 & 0 \\ 0 & 2 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2\mapsto R2/2} \begin{bmatrix} 1 & 0 & -0.5 & 0 \\ 0 & 1 & 5.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By solving the system of linear equations that is represented by the resulting row-reduced matrix we find that every vector of the form $\vec{\mathbf{x}} = \begin{bmatrix} 0.5x_3 \\ -5.5x_3 \\ x_3 \end{bmatrix}$ with $x_3 \neq 0$ is an eigenvector of $\mathbf{C}$ with eigenvalue $\lambda_2 = 5$. In particular, $\vec{\mathbf{x}}_2 = \begin{bmatrix} 0.5 \\ -5.5 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_2 = 5$.

Now let us find an eigenvector with eigenvalue $\lambda_3 = -1$:

Form $\mathbf{C} - \lambda_3\mathbf{I} = \mathbf{C} + \mathbf{I} = \begin{bmatrix} 1+1 & 0 & 2 \\ 6 & 7+1 & 8 \\ 4 & 0 & 3+1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 6 & 8 & 8 \\ 4 & 0 & 4 \end{bmatrix}$

The system $(\mathbf{C}+\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$ can be written as
$$\begin{array}{ccccccc} 2x_1 & & & + & 2x_3 & = & 0 \\ 6x_1 & + & 8x_2 & + & 8x_3 & = & 0 \\ 4x_1 & & & + & 4x_3 & = & 0 \end{array}$$

Perform Gaussian elimination on the extended matrix of this system:

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 6 & 8 & 8 & 0 \\ 4 & 0 & 4 & 0 \end{bmatrix} \xrightarrow{R2\mapsto R2-3R1} \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 8 & 2 & 0 \\ 4 & 0 & 4 & 0 \end{bmatrix} \xrightarrow{R3\mapsto R3-2R1} \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 8 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1\mapsto R1/2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 8 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2\mapsto R2/8} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By solving the system of linear equations that is represented by the resulting row-reduced matrix we find that every vector of the form $\vec{\mathbf{x}} = \begin{bmatrix} -x_3 \\ -0.25x_3 \\ x_3 \end{bmatrix}$ with $x_3 \neq 0$ is an eigenvector of $\mathbf{C}$ with eigenvalue $\lambda_3 = -1$. In particular, $\vec{\mathbf{x}}_3 = \begin{bmatrix} -1 \\ -0.25 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{C}$ with eigenvalue $\lambda_3 = -1$.

By this procedure we have found the following maximal linearly independent set of eigenvectors of $\mathbf{C}$: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -5.5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -0.25 \\ 1 \end{bmatrix} \right\}$

Question 67.5: Find a maximal linearly independent set of eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} 8 & 3 \\ -6 & -1 \end{bmatrix}$ of Question 67.1 of Module 67A.

Question 67.6: Find a maximal linearly independent set of eigenvectors of the matrix

$\mathbf{B} = \begin{bmatrix} -10 & 10 \\ 0.1 & -0.1 \end{bmatrix}$ of Question 67.2 of Module 67A.

Question 67.7: Find a maximal linearly independent set of eigenvectors of the matrix

$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ of Question 67.3 of Module 67A.

Question 67.8: Find a maximal linearly independent set of eigenvectors of the matrix

$\mathbf{D} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 5 & 0 & 0 \\ 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ of Question 67.4 of Module 67A.