# MATH3200: APPLIED LINEAR ALGEBRA
# PRACTICE MODULE 68A: EIGENVECTORS AND EIGENVALUES OF INVERSE MATRICES AND OF MATRIX TRANSPOSES

WINFRIED JUST, OHIO UNIVERSITY

This module is based on Lecture 38 and Conversation 34. Recall the following theorems from Lecture 38:

**Theorem 1.** *Let* $\mathbf{A}$ *be an invertible matrix, and let* $\vec{\mathbf{x}}$ *be an eigenvector of* $\mathbf{A}$ *with eigenvalue* $\lambda$. *Then* $\vec{\mathbf{x}}$ *is an eigenvector of* $\mathbf{A}^{-1}$ *with eigenvalue* $\frac{1}{\lambda}$.

In other words, $\mathbf{A}$ and $\mathbf{A}^{-1}$ have the same eigenvectors, and the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$.

**Theorem 2.** *Let* $\mathbf{A}$ *be a square matrix. Then* $\mathbf{A}$ *and* $\mathbf{A}^T$ *have the same eigenvalues.*
*However,* $\mathbf{A}$ *and* $\mathbf{A}^T$ *may have different eigenvectors.*
*But if* $\vec{\mathbf{x}}$ *is an eigenvector of* $\mathbf{A}$ *with eigenvalue* $\lambda$*, then* $\vec{\mathbf{x}}^T$ *is a* left eigenvector *with eigenvalue* $\lambda$ *of* $\mathbf{A}^T$*, which means that* $\vec{\mathbf{x}}^T\mathbf{A}^T = \lambda\vec{\mathbf{x}}^T$.
*Similarly, if* $\vec{\mathbf{y}}$ *is a left eigenvector of* $\mathbf{A}$ *with eigenvalue* $\lambda$*, then* $\vec{\mathbf{y}}^T$ *is an eigenvector with eigenvalue* $\lambda$ *of* $\mathbf{A}^T$.

Note that Theorem 2 only asserts that $\mathbf{A}$ and $\mathbf{A}^T$ *may* have different eigenvectors, *not* that they *always* have different eigenvectors. For example, when $\mathbf{A}$ is symmetric, then $\mathbf{A}^T = \mathbf{A}$, and it follows that $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvectors in this case.

**Question 68.1:** Suppose $\mathbf{A}$ is a $2 \times 2$ matrix such that $\mathbf{A}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \end{bmatrix}$ and $\mathbf{A}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

(a) What can you deduce about the eigenvectors and eigenvalues of $\mathbf{A}^{-1}$?
(b) What can you deduce about the eigenvectors and eigenvalues of $\mathbf{A}^T$?

**Question 68.2:** Suppose $\mathbf{A}$ is a $2 \times 2$ matrix such that $\mathbf{A}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{A}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

(a) What can you deduce about the eigenvectors and eigenvalues of $\mathbf{A}^{-1}$?
(b) What can you deduce about the eigenvectors and eigenvalues of $\mathbf{A}^T$?

**Question 68.3:** Suppose $\mathbf{A}$ is a $2 \times 2$ matrix such that $\mathbf{A}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \end{bmatrix}$ and $\mathbf{A}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \end{bmatrix}$

(a) What can you deduce about the eigenvectors and eigenvalues of $\mathbf{A}^{-1}$?
(b) What can you deduce about the eigenvectors and eigenvalues of $\mathbf{A}^T$?

**Question 68.4:** Suppose $\mathbf{A}$ is a $2 \times 2$ matrix such that $\mathbf{A}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \end{bmatrix}$ and $[2, 4]\mathbf{A} = [1, 2]$.

What can you deduce about the eigenvectors and left eigenvectors of $\mathbf{A}^T$?

**Question 68.5:** Prove that if $\vec{\mathbf{x}}$ is a left eigenvector with eigenvalue $\lambda$ of an invertible square matrix $\mathbf{A}$, then $\vec{\mathbf{x}}$ is a left eigenvector of $\mathbf{A}^{-1}$ with eigenvalue $\frac{1}{\lambda}$.
*Hint:* Adapt Alice's proof of the analogous result for eigenvectors from Conversation 34.