MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 68B: FINDING EIGENVALUES AND EIGENVECTORS WITH MATLAB

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This module is based on Lecture 38. It also quotes results from Lecture 36 and Module 67B. The module also uses MatLab. Start a MatLab session now.

Recall the following theorems from Lecture 38:

Theorem 1. *Let* $\mathbf{A}$ *be an invertible matrix, and let* $\vec{\mathbf{x}}$ *be an eigenvector of* $\mathbf{A}$ *with eigenvalue* λ. *Then* $\vec{\mathbf{x}}$ *is an eigenvector of* $\mathbf{A}^{-1}$ *with eigenvalue* $\frac{1}{\lambda}$.

In other words, $\mathbf{A}$ and $\mathbf{A}^{-1}$ have the same eigenvectors, and the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$.

Theorem 2. *Let* $\mathbf{A}$ *be a square matrix. Then* $\mathbf{A}$ *and* $\mathbf{A}^T$ *have the same eigenvalues.*
However, $\mathbf{A}$ *and* $\mathbf{A}^T$ *may have different eigenvectors.*
But if $\vec{\mathbf{x}}$ *is an eigenvector of* $\mathbf{A}$ *with eigenvalue* λ*, then* $\vec{\mathbf{x}}^T$ *is a* left eigenvector *with eigenvalue* λ *of* $\mathbf{A}^T$*, which means that* $\vec{\mathbf{x}}^T\mathbf{A}^T = \lambda\vec{\mathbf{x}}^T$.
Similarly, if $\vec{\mathbf{y}}$ *is a left eigenvector of* $\mathbf{A}$ *with eigenvalue* λ*, then* $\vec{\mathbf{y}}^T$ *is an eigenvector with eigenvalue* λ *of* $\mathbf{A}^T$.

Consider the matrix $\mathbf{B} = \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$ that we studied in Lecture 36.

We can compute the eigenvectors and eigenvalues of $\mathbf{B}$ in MatLab by entering:

```
>> B = [8, -6; 3, -1]
>> [vec, val] = eig(B)
```

We get two matrices as outputs:

`>> vec =` $\begin{matrix} 0.8944 & 0.7071 \\ 0.4472 & 0.7071 \end{matrix}$

`>> val =` $\begin{matrix} 5 & 0 \\ 0 & 2 \end{matrix}$

As usual, MatLab doesn't put square brackets around output matrices. The meaning of the matrix `val` is easy to understand: It has the eigenvalues of the input matrix on the (main) diagonal. These are the numbers $\lambda_1 = 5$ and $\lambda_2 = 2$ that we found as eigenvalues of $\mathbf{B}$ in Lecture 36. The matrix `vec` contains eigenvectors, but it is more difficult to interpret. The vector in column i of `vec` is an eigenvector with the eigenvalue that appears in column i on the diagonal of `val`. There are infinitely many such eigenvectors, and MatLab habitually chooses examples with coordinates that don't look particularly intuitive. We will discuss the reason for this in Chapter 5. Sometimes it is possible to rescale them to nicer-looking ones by dividing each of MatLab's choice of the eigenvectors by one of its nonzero coordinates. Try:

```
>> vec(:,1)
>> ans/ans(2)
>> vec(:,2)
>> ans/ans(2)
```

You will see the eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ of $\mathbf{B}$ that we found in Lecture 36.

Question 68.6: Let $\mathbf{A} = \begin{bmatrix} 628 & 77 \\ 198 & -43 \end{bmatrix}$ Use MatLab to find the eigenvalues of $\mathbf{A}$ and

(a) two linearly independent eigenvectors of $\mathbf{A}$ with integer coordinates,
(b) two linearly independent left eigenvectors of $\mathbf{A}$ with integer coordinates.

Hint: For part (b), notice that $\mathbf{A} = (\mathbf{A}^T)^T$ and that therefore the left eigenvectors of $\mathbf{A}$ are the transposes of the eigenvectors of $\mathbf{A}^T$. Recall also that MatLab uses `A'` for the transpose of a matrix `A`.

Now let us see what happens when we try to calculate the eigenvalues and eigenvectors of two matrices that do not have full sets of eigenvectors.

```
>> B = [0, -1; 1, 0]
>> [vec, val] = eig(B)
```

Here MatLab reports scaled versions of the complex eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$ with complex eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$ that we explored in Module 67B.

Recall that in Module 67B we found that all eigenvectors of the matrix $\begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$ are of the form $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$ for some $x_1 \neq 0$. Let's try to confirm this:

```
>> C = [5, 1, 0; 0, 5, 1; 0, 0, 5 ]
>> [vec, val] = eig(C)
```

Here MatLab will output three slightly different-looking eigenvectors with eigenvalue $\lambda = 5$, but they all lie on the same line through the origin, so they are linearly dependent.