MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY MODULE 8: ENTERING MATRICES IN MATLAB AND PROPERTIES OF MATRIX MULTIPLICATION

WINFRIED JUST, OHIO UNIVERSITY

This module studies several important properties of matrix multiplication and of the transpose that are not all explicitly covered in the lectures. For easier reference, let us list them here:

- (Associativity Law) If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices such that both $\mathbf{AB}$ and $\mathbf{BC}$ are defined, then $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.
- (Left Distributivity Law) If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices such that all relevant operations are defined, then $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.
- (Right Distributivity Law) If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices such that all relevant operations are defined, then $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$.
- Failure of Commutativity Even when $\mathbf{AB}$ and $\mathbf{BA}$ both exist and have the same order, we **may** have $\mathbf{AB} \neq \mathbf{BA}$.
- Let $\mathbf{A}$ be any matrix. Then $(\mathbf{A}^T)^T = \mathbf{A}$.
- Let $\mathbf{A}, \mathbf{B}$ be any matrices whose elements are numbers such that $\mathbf{AB}$ is defined.
  Then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

This module uses MatLab. You want to work through it while running a MatLab session. Start one now.

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 5 & -4 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 10 & 2 \\ -2 & 1 \end{bmatrix}$

To create these matrices in MatLab's **Command Window** enter:

```
>> A = [1, 2, 3; 4, 5, 6]
>> B = [-1 0; 5 -4]
>> C = [10,2;-2,1]
```

Note that elements of rows can be separated by commas, blank spaces, or both; columns need to be separated by semicolons.

For some special types of matrices there are shortcuts. To create a 1×2 row vector of 1s and a 3×1 column vector of 0s, enter:

```
>> r2Ones = ones(1,2)
>> cZ3 = zeros(3,1)
```

Note that we have given our vectors here fancy names so that we can remember more easily what they are. Most character strings of letters and numbers can be used for naming variables and functions in MatLab.

Now let's try to multiply some of these matrices and see what happens. Enter:

```
>> AB
```

MatLab will complain about an `Undefined function or variable 'AB'.` And rightly so: We *could* have defined a matrix named `AB`. We didn't, but MatLab thinks we might have wanted to. We used the wrong syntax. What we should have entered is:

```
>> A*B
```

Again MatLab will complain and point out to us that the `Inner matrix dimensions must agree.` Fair enough, $\mathbf{A}$ is a 2×3-matrix and $\mathbf{B}$ is a 2×2 matrix. Since $3 \neq 2$, the matrix product $\mathbf{AB}$ is undefined. Let's try another product:

```
>> B*A
```

This time we get an answer. We can multiply it with our column vector of 0's:

```
>> ans*cZ3
```

If we would like to do this later in the session, we would presumably want to enter:

```
>> (B*A)*cZ3
```

We get the same result. Now try:

```
>> A*cZ3
>> B*ans
>> B*(A*cZ3)
```

Again we get the same answer. We could even omit the brackets altogether here:

```
>> B*A*cZ3
```

Do we always get the same result because the vector `cZ3` is special, or because the brackets don't matter here? The *Associativity Law* of matrix multiplication assures us that brackets don't matter here:

- (Associativity Law) If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices such that both $\mathbf{AB}$ and $\mathbf{BC}$ are defined, then $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

Brackets do matter though if we mix matrix addition and matrix multiplication. Try:

```
>> B + C*C
>> (B + C)*C
```

For combining matrix addition with matrix multiplication, the following *Distributivity Laws* hold:

- (Left Distributivity Law) If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices such that all relevant operations are defined, then $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.
- (Right Distributivity Law) If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices such that all relevant operations are defined, then $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$.

Let's test the Right Distributivity Law for our matrices. Enter:

```
>> (B + C)*A
>> B*A + C*A
```

MatLab's answers will be the same. This gives an example, but not a formal proof of the law, which is supposed to hold for all matrices that permit the relevant operations. It is not even a formal proof of the law for all 2×2 matrices.

Question 8.1: Formally prove the Left Distributivity Law under the additional assumption that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices of order 2×2 whose elements are numbers.

How about commutativity? We have already seen an example where only one of the products $\mathbf{AB}, \mathbf{BA}$ is defined. But what if they are both defined? Must they then be equal? Try:

```
>> B*C
>> C*B
```

MatLab gives us different answers. We have shown that

- Failure of Commutativity Even when $\mathbf{AB}$ and $\mathbf{BA}$ both exist and have the same order, we *may* have $\mathbf{AB} \neq \mathbf{BA}$.

Question 8.2: Give examples of matrices $\mathbf{A}, \mathbf{B}$ such that $\mathbf{AB}$ and $\mathbf{BA}$ both exist but do not have the same order.

Before we continue our theoretical exploration of properties, let's take a break and do something useful with the vector `r2Ones` that we have created.

Question 8.3: Use MatLab to calculate the products `r2Ones*A, r2Ones*B, r2Ones*C`. What do these products represent?

We saw that commutativity of matrix multiplication *may* fail, but it *will not always fail.*

Let $\quad \mathbf{D} = \begin{bmatrix} -1 & 5 \\ 5 & -1 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}$

Enter these matrices in MatLab with variable names `D` and `E`, and have it compute the products `D*E` and `E*D`. You should get identical answers, which shows that $\mathbf{DE} = \mathbf{ED}$.

The matrices $\mathbf{D}$ and $\mathbf{E}$ of the above example look rather special (we will come back to this point in a moment). When we multiply arbitrarily chosen matrices $\mathbf{A}, \mathbf{B}$, say of order 2×2, will the equality $\mathbf{AB} = \mathbf{BA}$ *usually* hold, or will it *usually* fail? MatLab has a neat tool for exploring this question by letting us create *random* or *typical* matrices of any given order. Try:

```
>> R1 = rand(2)
>> R2 = rand(2)
>> R1*R2
>> R2*R1
```

The first two commands create random matrices of order 2×2 with entries in the interval $[0, 1]$. When you multiply them in different order, with very, very high probability you will get different results. You may want to repeat the above four commands a few times to convince yourself that this is indeed the case. It turns out that the equality $\mathbf{AB} = \mathbf{BA}$ *typically fails* for matrix multiplication.

However, commutativity may actually hold if we restrict ourselves to multiplication of very special kinds of matrices. Take a second look at the matrices $\mathbf{D}$ and $\mathbf{E}$ above. They are very special indeed: Each of them is symmetric and has two identical elements on the (main) diagonal. Multiplication of two such matrices is commutative.

Question 8.4: Let $\mathbf{A} = [a_{ij}]_{2\times 2}$ and $\mathbf{B} = [b_{ij}]_{2\times 2}$ be two symmetric 2×2 matrices of numbers such that $a_{11} = a_{22}$ and $b_{11} = b_{22}$. Prove that $\mathbf{AB} = \mathbf{BA}$.

Would it be sufficient in Question 8.4 to assume only that $\mathbf{A}$ an $\mathbf{B}$ are of order 2×2 and symmetric? Let's explore this problem in MatLab first. Enter:

```
>> E(2,2) = 6
```

This changes the element in the second row and second column of the matrix `E` to `6`. This new matrix is still symmetric, but it has two different elements on its main diagonal. Now let's calculate the products of this new version of `E` with `D`:

```
>> D*E
>> E*D
```

You should see different answers, and we can conclude that we could not prove a result as in Question 8.4 by assuming only symmetry. Accidentally, these calculations also show that the product of two symmetric square matrices need not be symmetric.

But now take another look at the latest MatLab output: The two matrix products are different, but the second one is the transpose of the first. Is this an accident, or does it indicate a general pattern?

We already know how we can explore this question in MatLab: By multiplying random symmetric matrices. Enter:

```
>> R1 = rand(2)
>> R1(2,1) = R1(1,2)
```

The last line here creates a symmetric 2×2 matrix by setting the element in the first row and second column equal to the other off-diagonal element of `R1`.

```
>> R2 = rand(2)
>> R2(2,1) = R2(1,2)
>> R1*R2
>> R2*R1
```

Does it appear that there is a pattern here?

Question 8.5: Let $\mathbf{A}$ and $\mathbf{B}$ be two symmetric 2×2 matrices of numbers. Prove that $\mathbf{AB}$ and $\mathbf{BA}$ are transposes of each other.

This brings us to our next question: *How does matrix multiplication behave with respect to matrix transposes?* Let's begin with the relevant syntax. To create the transpose $\mathbf{A}^T$ in MatLab, enter:

```
>> A'
```

Now try:

```
>> ans'
```

and repeat a few times to toggle between `A` and its transpose. This illustrates an important property of the matrix transpose:

- Let $\mathbf{A}$ be any matrix. Then $(\mathbf{A}^T)^T = \mathbf{A}$.

Now let us explore how the transpose $(\mathbf{AB})^T$ of a matrix product is related to the product of the transposes $\mathbf{A}^T$ and $\mathbf{B}^T$. One might suspect that $(\mathbf{AB})^T = \mathbf{A}^T\mathbf{B}^T$. Let's try it with some of our examples that we are actually allowed to multiply:

```
>> (B*C)'
>> B'*C'
>> (B*A)'
>> B'*A'
```

No luck with this one. But now let us try *reversing* the order of the transposes of the factors:

```
>> (B*C)'
>> C'*B'
>> (B*A)'
>> A'*B'
```

This time we do get the same matrices. This illustrates the following property:

- Let $\mathbf{A}, \mathbf{B}$ be any matrices of numbers such that the product $\mathbf{AB}$ is defined.
  Then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Note that while the previous properties hold for multiplication of matrices with arbitrary elements, such as matrices of matrices, here we assume that the elements of the matrices are numbers. Let's prove this property at least for matrices of order 2×2.

Question 8.6: Prove that for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ of numbers the equality $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ holds.

Question 8.7: Prove that for every 2×2 square matrix $\mathbf{A}$, if the sum $\mathbf{A} + \mathbf{A}^T$ is defined, then it is a symmetric matrix.