MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 81: NORMS AND DISTANCES

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This module is based on Lecture 41. Recall the following notions from this lecture:

The *norm* of a vector $\vec{\mathbf{x}}$ is a real number $\|\vec{\mathbf{x}}\|$ that can be thought of as its length. Formally norm is a function $\|\cdot\|$ defined on a given vector space that has the following properties:

(i) $\|\vec{\mathbf{x}}\| \geq 0$ and $\|\vec{\mathbf{x}}\| = 0$ if, and only if, $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
(ii) If α is any scalar, then $\|\alpha\vec{\mathbf{x}}\| = |\alpha|\,\|\vec{\mathbf{x}}\|$.
(iii) $\|\vec{\mathbf{x}} + \vec{\mathbf{y}}\| \leq \|\vec{\mathbf{x}}\| + \|\vec{\mathbf{y}}\|$ for any vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$.

A *unit vector* wrt (with respect to) a given norm $\|\cdot\|$ is a vector $\vec{\mathbf{x}}$ such that $\|\vec{\mathbf{x}}\| = 1$.
The *normalization* of $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ wrt a given norm $\|\cdot\|$ is the vector $\frac{\vec{\mathbf{x}}}{\|\vec{\mathbf{x}}\|}$.
The *distance* $d(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ between vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ (wrt a given norm $\|\cdot\|$) is defined as
$d(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = \|\vec{\mathbf{x}} - \vec{\mathbf{y}}\|$.

Remarks on the notation: In the expression $\|\cdot\|$ the dot is a placeholder for the input of the function. Thus we would write $\|\cdot\|$ if we want to talk about the norm as a function and $\|\vec{\mathbf{x}}\|$ if we want to consider the norm of a given vector $\vec{\mathbf{x}}$. In the lectures for this chapter the symbol $\|\cdot\|$ will always denote the familiar *Euclidean norm* on a given space $\mathbb{R}^n$ that is defined as
$\|\vec{\mathbf{x}}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

However, there are many different norms on a given vector space $\mathbb{R}^n$ and in this module we will use subscripts to distinguish between them. In particular, for any real number $p \geq 1$ one can define a norm $\|\cdot\|_p$ on $\mathbb{R}^n$ by
$\|\vec{\mathbf{x}}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$.
In this notation, the Euclidean norm is the special case for $p = 2$, and *in this module the Euclidean norm will be denoted by* $\|\cdot\|_2$. Another special case that we will consider here is the norm that we obtain by setting $p = 1$:
$\|\vec{\mathbf{x}}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$.
Yet another norm of interest is defined as
$\|\vec{\mathbf{x}}\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$.
The reason why we use the subscript ∞ for the last norm is that for any given vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ it is true that $\lim_{p\to\infty} \|\vec{\mathbf{x}}\|_p = \max\{|x_1|, |x_2|, \ldots, |x_n|\} = \|\vec{\mathbf{x}}\|_\infty$.
For example, consider the vector $\vec{\mathbf{x}} = [1, 2, -2]$ in $\mathbb{R}^3$. In Lecture 41 we have seen that
$\|[1, 2, -2]\|_2 = \sqrt{1^2 + 2^2 + (-2)^2)} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$.
For the other two norms that we consider here we get
$\|[1, 2, -2]\|_1 = |1| + |2| + |-2| = 1 + 2 + 2 = 5$ and $\|[1, 2, -2]\|_\infty = \max\{|1|, |2|, \ldots, |-2|\} = 2$.

Similarly, the values of the distance function depend on the particular norm that was used in its definition. Consider, for example, $\vec{\mathbf{x}} = [3, 4, 5]$ and $\vec{\mathbf{y}} = [2, 2, 7]$. Then $\vec{\mathbf{x}} - \vec{\mathbf{y}} = [1, 2, -2]$, and in Lecture 41 we found that $d([3, 4, 5], [2, 2, 7]) = \|[1, 2, -2]\|_2 = 3$ if the distance is based on the Euclidean norm $\|\cdot\|_2$. It follows from the above that $d([3, 4, 5], [2, 2, 7]) = \|[1, 2, -2]\|_1 = 5$ if the distance is based on the norm $\|\cdot\|_1$ and $d([3,4,5],[2,2,7]) = \|[1,2,-2]\|_\infty = 2$ if the distance is based on the norm $\|\cdot\|_\infty$.

On can think of using different norms as analogous to measuring lengths in different units: centimeters, meters, inches, yards, etc. The Euclidean norm corresponds to our usual notion of length and it works best for most, but not for all applications. For example, consider driving in Manhattan. There you can move only either south/north or east/west along any city block, and the driving distance between two intersections would could be calculated in terms of the norm $\|\cdot\|_1$ instead of the Euclidean norm $\|\cdot\|_2$. Similarly, in applications to Markov chains we may want to use the norm $\|\vec{\mathbf{x}}\|_1$, as for this norm all vectors that represent probability distributions are unit vectors with respect to $\|\vec{\mathbf{x}}\|_1$.

Question 81.1: Let $\vec{\mathbf{x}} = [-3, 4]$.

(a) Find $\|\vec{\mathbf{x}}\|_1$.
(b) Find $\|\vec{\mathbf{x}}\|_2$.
(c) Find $\|\vec{\mathbf{x}}\|_\infty$.

Question 81.2: Let $\vec{\mathbf{x}} = [0.5, -0.2, -0.3]$. With respect to which among the norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ is $\vec{\mathbf{x}}$ a unit vector?

Question 81.3: Let $\vec{\mathbf{x}} = [-4, 3]$.

(a) Normalize $\vec{\mathbf{x}}$ wrt $\|\cdot\|_1$, that is, find the normalization of $\vec{\mathbf{x}}$ wrt $\|\cdot\|_1$.
(b) Normalize $\vec{\mathbf{x}}$ wrt $\|\cdot\|_2$, that is, find the normalization of $\vec{\mathbf{x}}$ wrt $\|\cdot\|_2$.
(c) Normalize $\vec{\mathbf{x}}$ wrt $\|\cdot\|_\infty$, that is, find the normalization of $\vec{\mathbf{x}}$ wrt $\|\cdot\|_\infty$.

Question 81.4: Let $\vec{\mathbf{x}} = [5, 4, 3, 2, 1]$ and $\vec{\mathbf{y}} = [4, 6, 2, 3, 4]$. Find the distance $d(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ if

(a) The distance is defined wrt the norm $\|\cdot\|_1$.
(b) The distance is defined wrt the norm $\|\cdot\|_2$.
(c) The distance is defined wrt the norm $\|\cdot\|_\infty$.