MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 82B: PROPERTIES OF INNER PRODUCTS

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This module is based on Lecture 42. Recall the following definitions and facts from this lecture:

The (standard) *inner product* $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle$ of vectors $\vec{\mathbf{x}} = [x_1, x_2, \dots, x_n], \vec{\mathbf{y}} = [y_1, y_2, \dots, y_n]$ is defines as $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n = \sum_{i=1}^{n} x_iy_i$.

Theorem 1. *Let $\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}}$ be any vectors of the same order, and let λ be a scalar. Then*
(I1) $\langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle \geq 0$. Moreover, $\langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle = 0$ if, and only if, $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
(I2) $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = \langle \vec{\mathbf{y}}, \vec{\mathbf{x}} \rangle$.
(I3) $\langle \lambda\vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = \lambda\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = \langle \vec{\mathbf{x}}, \lambda\vec{\mathbf{y}} \rangle$.
(I4) $\langle \vec{\mathbf{x}} + \vec{\mathbf{z}}, \vec{\mathbf{y}} \rangle = \langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle + \langle \vec{\mathbf{z}}, \vec{\mathbf{y}} \rangle$ and $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} + \vec{\mathbf{z}} \rangle = \langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle + \langle \vec{\mathbf{x}}, \vec{\mathbf{z}} \rangle$.
(I5) $\langle \vec{\mathbf{0}}, \vec{\mathbf{y}} \rangle = \langle \vec{\mathbf{x}}, \vec{\mathbf{0}} \rangle = 0$.

In a fully general development of linear algebra any function $\langle \cdot, \cdot \rangle$ of two variables that satisfies the above properties would be called "an inner product." While there many important applications of nonstandard inner products, in this course we will not go to this level of abstraction and confine ourselves to he standard one. It is however of interest to explore why our standard inner product has these properties; this is what we will do here.

Each part of Theorem 1 can be proved for the standard inner product directly from its definition. We can focus on row vectors; the proof for column vectors will essentially be the same.

For Properties (I1)–(I3), we can argue as follows:

Let $\vec{\mathbf{x}} = [x_1, x_2, \dots, x_n]$, $\vec{\mathbf{y}} = [y_1, y_2, \dots, y_n]$ be any row vectors of length n, and let λ be a scalar. Then $\langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle = x_1x_1 + x_2x_2 + \dots + x_nx_n = x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$.
Moreover, $\langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle = 0$ if, and only if, $x_1 = x_2 = \dots = x_n = 0$, that is, if $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
This proves Property (I1).

$\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n = y_1x_1 + y_2x_2 + \dots + y_nx_n = \langle \vec{\mathbf{y}}, \vec{\mathbf{x}} \rangle$. This proves Property (I2).

$$\begin{aligned}\langle \lambda\vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle &= \langle [\lambda x_1, \lambda x_2, \dots \lambda x_n], [y_1, y_2, \dots, y_n] \rangle = \lambda x_1y_1 + \lambda x_2y_2 + \dots + \lambda x_ny_n \\ &= \lambda(x_1y_1 + x_2y_2 + \dots + x_ny_n) \\ &= \lambda\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle.\end{aligned}$$

This proves the first part of Property (I3); the proof of the second part is analogous.

Question 82.5: Prove that the standard inner product has Properties (I4) and (I5).

Question 82.6: How is $\langle \lambda\vec{\mathbf{x}}, \lambda\vec{\mathbf{y}} \rangle$ related to $\langle \vec{\mathbf{x}}, \vec{\mathbf{y}} \rangle$?