MATH3200: APPLIED LINEAR ALGEBRA
PRACTICE MODULE 83: PROJECTIONS, ORTHOGONAL COMPLEMENTS, AND ORTHONORMAL BASES

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This module is based on Lecture 43. Recall the following facts from this lecture:

Theorem 1. *Let $\vec{\mathbf{a}}, \vec{\mathbf{x}}$ be two vectors in $\mathbb{R}^n$.*
Then there exists exactly one pair $(\vec{\mathbf{u}}, \vec{\mathbf{v}})$ of vectors in $\mathbb{R}^n$ such that

(1) $\vec{\mathbf{x}} = \vec{\mathbf{u}} + \vec{\mathbf{v}}$.
(2) $\vec{\mathbf{u}} = c\vec{\mathbf{a}}$ *for some scalar c, that is, $\vec{\mathbf{u}}, \vec{\mathbf{a}}$ are on the same line.*
(3) $\langle \vec{\mathbf{a}}, \vec{\mathbf{v}} \rangle = \langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = 0$, *that is, $\vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{a}}$ and $\vec{\mathbf{u}}$.*

When $\vec{\mathbf{a}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ and $\vec{\mathbf{v}} = \vec{\mathbf{x}}$. When $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$, then

$$\vec{\mathbf{u}} = \frac{\langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{a}}, \vec{\mathbf{a}} \rangle} \vec{\mathbf{a}} \qquad \text{and} \qquad \vec{\mathbf{v}} = \vec{\mathbf{x}} - \vec{\mathbf{u}} = \vec{\mathbf{x}} - \frac{\langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{a}}, \vec{\mathbf{a}} \rangle} \vec{\mathbf{a}}.$$

The vector $\vec{\mathbf{u}}$ is called the *projection of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{a}}$*, and $\vec{\mathbf{v}}$ is called the *orthogonal complement of $\vec{\mathbf{x}}$ with respect to $\vec{\mathbf{a}}$.*

A set $B = \{\vec{\mathbf{b}}_1, \dots, \vec{\mathbf{b}}_n\}$ of vectors is *orthogonal* if $\langle \vec{\mathbf{b}}_i, \vec{\mathbf{b}}_j \rangle = 0$ for all $i \neq j$.

We call B *orthonormal* if it is orthogonal and composed of unit vectors, so that $\|\vec{\mathbf{b}}_i\| = \langle \vec{\mathbf{b}}_i, \vec{\mathbf{b}}_i \rangle = 1$ for all $i = 1, \dots, n$.

A set B is an *orthonormal basis* of a vector space V if it is orthonormal and a basis for V.

$B = \{\vec{\mathbf{e}}_1, \dots, \vec{\mathbf{e}}_n\}$ is an example of an orthonormal basis of $\mathbb{R}^n$.

When B is an orthonormal basis of a vector space V, then every vector $\vec{\mathbf{x}}$ can be written in the alternative coordinates $\vec{\mathbf{c}}$ with respect to B that are given by
$\vec{\mathbf{c}} = [\langle \vec{\mathbf{b}}_1, \vec{\mathbf{x}} \rangle, \langle \vec{\mathbf{b}}_2, \vec{\mathbf{x}} \rangle, \dots, \langle \vec{\mathbf{b}}_n, \vec{\mathbf{x}} \rangle]$.

Let $\vec{\mathbf{x}} = [1, 1]$, $\vec{\mathbf{y}} = [1, -1]$, $\vec{\mathbf{z}} = [5, 0]$.

Question 83.1: (a) Find the projection $\vec{\mathbf{u}}$ of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{y}}$.
(b) Find the orthogonal complement $\vec{\mathbf{v}}$ of $\vec{\mathbf{x}}$ with respect to $\vec{\mathbf{y}}$.

Question 83.2: (a) Find the projection $\vec{\mathbf{u}}$ of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{z}}$.
(b) Find the orthogonal complement $\vec{\mathbf{v}}$ of $\vec{\mathbf{x}}$ with respect to $\vec{\mathbf{z}}$.

Question 83.3: (a) Find the projection $\vec{\mathbf{u}}$ of $\vec{\mathbf{y}}$ onto $\vec{\mathbf{z}}$.
(b) Find the orthogonal complement $\vec{\mathbf{v}}$ of $\vec{\mathbf{y}}$ with respect to $\vec{\mathbf{z}}$.

Question 83.4: (a) Find the projection $\vec{\mathbf{u}}$ of $\vec{\mathbf{z}}$ onto $\vec{\mathbf{y}}$.
(b) Find the orthogonal complement $\vec{\mathbf{v}}$ of $\vec{\mathbf{z}}$ with respect to $\vec{\mathbf{y}}$.

Let $\vec{\mathbf{b}}_1 = \left[\frac{3}{5}, \frac{4}{5}\right]$ and $\vec{\mathbf{b}}_2 = \left[\frac{-4}{5}, \frac{3}{5}\right]$, and consider the set $B = \{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2\}$.

Question 83.5: Show that the set B is an orthonormal basis of $\mathbb{R}^2$.

Question 83.6: Use the fact that B is an orthonormal basis of $\mathbb{R}^2$ to express $\vec{\mathbf{e}}_1$ in alternative coordinates wrt B. That is, find $\vec{\mathbf{c}} = [c_1, c_2]$ such that $\vec{\mathbf{e}}_1 = c_1\vec{\mathbf{b}}_1 + c_2\vec{\mathbf{b}}_2$.