MATH3200: APPLIED LINEAR ALGEBRA SELF-STUDY AND PRACTICE MODULE 9: SUBMATRICES AND POWERS OF SQUARE MARICES

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We will use the terminology and notation of Lectures 5 and 6.

This module uses MatLab. You want to work through it while running a MatLab session. Start one now.

1. Self-Study and Practice: Submatrices in MatLab

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$

Enter this matrix in MatLab. This can be done in a reasonably efficient way as follows:

```
>> A = [1:5; 6:10; 11:15; 16:20; 21:25]
```

Here `1:5` is MatLab's shorthand for the vector `[1,2,3,4,5]`.

You can access the second column and the third row of `A` as follows:

```
>> A(:, 2)
>> A(3, :)
```

Let's make a copy of `A` and create a submatrix by removing the second column of as follows:

```
>> B = A
>> B(:, 2) = []
```

Here removal is accomplished by setting the second column of `B` to the *empty matrix* `[]`.

Let's create an even smaller submatrix by also removing the fourth and fifth rows:

```
>> B([4,5], :) = []
```

You could have created the same submatrix by entering:

```
>> B = A([1, 2, 3], [1, 3, 4, 5])
```

or by entering:

```
>> B = A([1, 2, 3], [1, 3, 4, 5])
```

Note that we are using here essentially the same syntax `A(i,j)` for accessing an element of `A` when `i` and `j` are positive integers for accessing an entire submatrix when we let `i` and `j` take values that are vectors.

Question 9.1: Let $\mathbf{B}_1 = \begin{bmatrix} 6 & 7 & 10 \\ 16 & 17 & 20 \end{bmatrix}$

(a) Is $\mathbf{B}_1$ a submatrix of the matrix $\mathbf{A}$ above?
(b) If so, what MatLab commands could be used to obtain it from `A`?

Question 9.2: Let $\mathbf{B}_2 = \begin{bmatrix} 2 & 3 & 5 \\ 16 & 17 & 19 \end{bmatrix}$

(a) Is $\mathbf{B}_2$ a submatrix of the matrix $\mathbf{A}$ above?
(b) If so, whatMATLAB commands could be used to obtain it from `A`?

Question 9.3: Let $\mathbf{B}_3 = \begin{bmatrix} 1 & 3 & 2 \\ 6 & 8 & 7 \end{bmatrix}$

(a) Is $\mathbf{B}_3$ a submatrix of the matrix $\mathbf{A}$ above?
(b) If so, what MATLAB commands could be used to obtain it from `A`?

Question 9.4: Let $\mathbf{B}_4 = \begin{bmatrix} 13 \end{bmatrix}$

(a) Is $\mathbf{B}_4$ a submatrix of the matrix $\mathbf{A}$ above?
(b) If so, what MATLAB commands could be used to obtain it from `A`?

Question 9.5: Let $\mathbf{B}_5 = \begin{bmatrix} 6 & 7 & 8 & 9 & 10 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$

(a) Is $\mathbf{B}_5$ a submatrix of the matrix $\mathbf{A}$ above?
(b) If so, what MATLAB commands could be used to obtain it from `A`?

2. Self-Study and Practice: Powers of Square matrices

Continuing in your MATLAB session, enter:

```
>> A*A
>> A^2
```

The output will be identical for both commands. What you have observed here is that we can define positive integer powers of square matrices in pretty much the same way as for numbers. More precisely:

Definition 1. *Let* $\mathbf{A}$ *be a square matrix. For a positive integer* r*, the* r*-th power of* $\mathbf{A}$ *is defined as*

$$\mathbf{A}^r = \mathbf{A}\mathbf{A}\mathbf{A}\cdots\mathbf{A}\mathbf{A},$$

where the product on the right contains r *terms.*

By the Associativity Law, when $r = p + q$ and $p, q > 0$, we can group the product into p factors followed by q factors as follows:
$\mathbf{A}^r = \mathbf{A}^{p+q} = (\mathbf{A}\mathbf{A}\cdots\mathbf{A})(\mathbf{A}\mathbf{A}\cdots\mathbf{A}) = \mathbf{A}^p\mathbf{A}^q,$
which is exactly the same rule as $a^{p+q} = a^p a^q$ for powers of numbers.
In particular, $\mathbf{A}^{p+1} = \mathbf{A}^p\mathbf{A}^1 = \mathbf{A}^p\mathbf{A}$.
Similarly, the law $(\mathbf{A}^r)^p = \mathbf{A}^{rp}$ holds for square matrices in analogy with the law $(a^r)^p = a^{rp}$ for numbers.

To see how the last two rules work out in MATLAB, enter:

```
>> A^2
>> ans*A
>> A^3
>> ans^2
```

```
>> A^6
```

For the last two commands, the MatLab output will show 5 significant digits for each element of the matrix, preceded by th information that these are supposed to be multiplied by 10^{10}.

Question 9.6: Give an intuitive explanation why the law $(\mathbf{A}^r)^p = \mathbf{A}^{rp}$ holds for all square matrices and all positive integers $p, q > 0$.

In analogy with $a^0 = 1$ we also let $\mathbf{A}^0 = \mathbf{I}_n$ when $\mathbf{A}$ has order $n \times n$.
Then $\mathbf{A}^q = \mathbf{I}\mathbf{A}^q = \mathbf{A}^0\mathbf{A}^q = \mathbf{A}^{0+q}$ and $(\mathbf{A}^0)^p = \mathbf{I}^p = \mathbf{I} = \mathbf{A}^0 = \mathbf{A}^{0p}$, so this would work just fine.

For some—but not for all—square matrices $\mathbf{A}$, one can also meaningfully define $\mathbf{A}^r$ for integers $r < 0$. We will return to this topic in Chapter 2.

Let us mention an interesting difference between powers a^r of numbers and powers $\mathbf{A}^r$. When $a^r = 0$ for some positive integers r, then a must be zero. However, there are so-called *nilpotent* matrices, square matrices $\mathbf{A}$ such that $\mathbf{A}^r = \mathbf{O}$ for some positive integer r, other than zero matrices. Enter:

```
>> N = [0 1 2; 0 0 3; 0 0 0]
```

and let MatLab compute successive powers of this matrix.

Question 9.7: What do you observe?