MATH3200: APPLIED LINEAR ALGEBRA
SELF-STUDY AND PRACTICE MODULE 72: DIAGONALIZATION

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This module is based on Conversation 37. For some questions you will need to use certain inverse matrices that you can look up in Lectures 16 and 18.

1. Practice: Finding matrices with specified eigenvectors and eigenvalues

Recall from Conversation 37 that when B is a basis that consists of eigenvectors of a matrix $\mathbf{A}$, then $\mathbf{A} = \mathbf{BDB}^{-1}$, where $\mathbf{D}$ is the diagonal matrix that lists the respective eigenvalues of the vectors in B on the main diagonal in the same order as these eigenvectors are written as columns of $\mathbf{B}$. Such a matrix $\mathbf{D}$ is called a *diagonalization of* $\mathbf{A}$.

Question 72.1: Find a matrix $\mathbf{A}$ or order 2×2 that has eigenvectors $\vec{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{\mathbf{x}}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, respectively.

Question 72.2: Find a matrix $\mathbf{A}$ or order 3×3 that has eigenvectors

$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$ $\vec{\mathbf{x}}_2 = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}$ $\vec{\mathbf{x}}_3 = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}$ with eigenvalues $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 = -1$, respectively.

2. Self-Study and Practice: Similar and Diagonalizable Matrices

Recall the following definition and theorem from Conversation 37:

Definition 1. *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be two square matrices of the same order. Then we say that* $\mathbf{A}$ *and* $\mathbf{C}$ *are* similar *if there exists an invertible matrix* $\mathbf{B}$ *such that*

$$\mathbf{C} = \mathbf{B}^{-1}\mathbf{AB}. \tag{1}$$

A square matrix $\mathbf{A}$ *that is similar to a diagonal matrix is called* diagonalizable.

Theorem 1. *Let* $\mathbf{A}$ *be any matrix of order* $n \times n$. *Then* $\mathbf{A}$ *is diagonalizable if, and only if, it has a full set of eigenvectors.*

In Conversation 37 Alice had given Frank credit for proving one half of Theorem 1 by showing that if $\mathbf{A}$ has a full set of eigenvectors, then $\mathbf{A}$ is diagonalizable in the sense of Definition 1. But it is not entirely obvious that Frank has actually proved this. Recall that Frank, with a little help from the others, had shown that if $\mathbf{A}$ has a full set of eigenvectors $\{\vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_n\}$ and if $\mathbf{B}$ is the matrix whose columns are these eigenvectors, while $\mathbf{D}$ is the diagonal matrix that lists their corresponding eigenvalues on its (main) diagonal, then

$$\mathbf{A} = \mathbf{BDB}^{-1}. \tag{2}$$

But notice that (1) looks slightly different. Definition 1 will give us diagonalizability of $\mathbf{A}$ only if we have

$$\mathbf{D} = \mathbf{B}^{-1}\mathbf{AB}. \tag{3}$$

So we still need to convice ourselves that (2) and (3) are actually saying the same thing.

Question 72.3: Prove that if (2) is true, then (3) is also true.

One can show that it also works the other way around: If (3) is true, then (2) is also true. This makes sense, because if a matrix $\mathbf{A}$ is "similar" to a matrix $\mathbf{D}$, then $\mathbf{D}$ should also be similar to $\mathbf{A}$.

Let us conclude by giving a proof of the other half of Theorem 1 by showing that if $\mathbf{A}$ is diagonalizable, then $\mathbf{A}$ has a full set of eigenvectors.

Assume that $\mathbf{A}$ is a diagonalizable matrix, with $\mathbf{D} = \mathbf{B}^{-1}\mathbf{AB}$ being diagonal for some invertible matrix $\mathbf{B}$. Since $\mathbf{B}$ is invertible, its columns form a linearly independent set, and it suffices to show that this set consists of eigenvectors of $\mathbf{A}$.
We also know that (2) holds, so we calculate $\mathbf{A}\vec{\mathbf{x}}_i$ for each column of $\mathbf{B}$ as follows:
$\mathbf{A}\vec{\mathbf{x}}_i = (\mathbf{BDB}^{-1})\vec{\mathbf{x}} = \mathbf{BD}(\mathbf{B}^{-1}\vec{\mathbf{x}})$.
We recognize $\mathbf{B}^{-1}\vec{\mathbf{x}}$ as the formula that gives the alternative coordinates of $\vec{\mathbf{x}}_i$ with respect to the basis that consists of the columns of $\mathbf{B}$. As our protagonists discovered in Conversation 37, this implies that $\mathbf{B}^{-1}\vec{\mathbf{x}}_i = \vec{\mathbf{e}}_i$. So we get:
$\mathbf{A}\vec{\mathbf{x}}_i = (\mathbf{BDB}^{-1})\vec{\mathbf{x}} = \mathbf{BD}(\mathbf{B}^{-1}\vec{\mathbf{x}}) = \mathbf{BD}\vec{\mathbf{e}}_i = \mathbf{B}(\mathbf{D}\vec{\mathbf{e}}_i)$.
Since $\mathbf{D}$ was assumed a diagonal matrix, $\vec{\mathbf{e}}_i$ is an eigenvector of $\mathbf{D}$, and $\mathbf{D}\vec{\mathbf{e}}_i = \lambda_i\vec{\mathbf{e}}_i$, where λ_i is the i^{th} element on the (main) diagonal of $\mathbf{D}$. From properties of scalar multiplication we get:
$\mathbf{A}\vec{\mathbf{x}}_i = (\mathbf{BDB}^{-1})\vec{\mathbf{x}} = \mathbf{BD}(\mathbf{B}^{-1}\vec{\mathbf{x}}) = \mathbf{BD}\vec{\mathbf{e}}_i = \mathbf{B}(\mathbf{D}\vec{\mathbf{e}}_i) = \mathbf{B}(\lambda_i\vec{\mathbf{e}}_i) = \lambda_i(\mathbf{B}\vec{\mathbf{e}}_i)$.
We recognize $\mathbf{B}\vec{\mathbf{e}}_i$ as the formula that gives the Cartesian coordinates $\vec{\mathbf{x}}_i$ of the vector with alternative coordinates $\vec{\mathbf{e}}_i$, that is, gives the i^{th} column of $\mathbf{B}$:
$\mathbf{A}\vec{\mathbf{x}}_i = (\mathbf{BDB}^{-1})\vec{\mathbf{x}} = \mathbf{BD}(\mathbf{B}^{-1}\vec{\mathbf{x}}) = \mathbf{BD}\vec{\mathbf{e}}_i = \mathbf{B}(\mathbf{D}\vec{\mathbf{e}}_i) = \mathbf{B}(\lambda_i\vec{\mathbf{e}}_i) = \lambda_i(\mathbf{B}\vec{\mathbf{e}}_i) = \lambda_i\vec{\mathbf{x}}_i$.
We conclude that $\vec{\mathbf{x}}_i$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ_i. $\square$