MATH 3200: OUTLINE OF CHAPTER 2
SYSTEMS OF LINEAR EQUATIONS

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This chapter covers linear systems, the nature of their solution sets, and their matrix representations. It also introduces the inverse of a square matrix. You will learn methods for solving systems of linear equations, most notably, Gaussian elimination.

We will usually refer to specific items of the material as follows: L1 means Lecture 1, C2 means Conversation 2, and M3 means Module 3.

1. Concepts and Facts

1.1. Systems of linear equations and their solution sets (L11).

- The general form of a linear equation is in n variables is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b$$

  - For $n = 2$ variables, this equation defines a line in $\mathbb{R}^2$.
  - For $n = 3$ variables, this equation defines a plane in $\mathbb{R}^3$.
  - For n variables, this equation defines a *hyperplane* in $\mathbb{R}^n$.
- A *system of m linear equations in n variables* is an expression

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

  We assume that all a_{ij} and b_i are given scalar constants.

  When $b_1 = b_2 = \cdots = b_m = 0$, the system is *homogeneous.*
- A column vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ of numbers that satisfy all equations is *a solution* of the system (1). The set of all solutions is the *solution set* of the system.
  - The system is *consistent* if it has at least one solution.
  - Consistent systems may have either one or infinitely many solutions. In the latter case the system is *underdetermined* aka *underconstrained.*
  - The system is *inconsistent* aka *overdetermined* aka *overconstrained* if it has no solution.
  - Homogeneous systems are always consistent.
  - When the number of variables n of a system (1) exceeds the number of equations m, the system will be either underdetermined or inconsistent.

1.2. **Matrix representations of systems of linear equations (L12).**

- The *coefficient matrix* of the system (1) is $\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$
- (The *extended* aka *augmented* matrix of the system (1) is the matrix

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{bmatrix}$$

- The *matrix form* of the system (1) is $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

1.3. **Row echelon form, solving systems, back-substitution (C12, L13, C13).**

- (C12) Two systems of m linear equations in n variables are *equivalent* if they have the same solution sets.
- (C12) *Solving* a system means finding its solution set. It essentially boils down to transforming the system successively into simpler equivalent ones.
- (C12) A particularly simple form of matrices is the *row echelon form* Linear system whose extended matrices are in this form can be solved by *back-substitution.*
- (L13) A matrix is in *row echelon form,* or simply a *echelon form* if:
  - (R1) All *zero rows,* that is, rows with only zeros, appear below all *nonzero rows* when both types are present.
  - (R2) The first nonzero entry in any nonzero row is 1.
  - (R3) All elements in the same column below the first nonzero element of a nonzero row are 0.
  - (R4) The first nonzero element in a nonzero row appears in a column further to the right of the first nonzero element in any preceding row.

  Some textbooks do not require condition (R2) in this definition. We will sometimes say that a matrix that satisfies conditions (R1), (R3), and (R4) is in *generalized echelon form.*
- (L13) A matrix is in *reduced row echelon form* or simply *reduced echelon form,* if:
  - (R1) All *zero rows,* that is, rows with only zeros, appear below all *nonzero rows* when both types are present.
  - (R2) The first nonzero entry in any nonzero row is 1.
  - (R3+) All elements in the same column as the first nonzero element of a nonzero row are 0.
  - (R4) The first nonzero element in a nonzero row appears in a column further to the right of the first nonzero element in any preceding row.

  Note that the only difference between the definitions of echelon form and reduced echelon form of a matrix is that we replaced the word "below" in the definition of condition (R3) with the word "as" in condition (R3+).
- (C13) Linear systems can be solved by first transforming them into equivalent systems whose matrix is in row echelon form by successively using the following *elementary operations on the equations* that do not change the solution set:

(i) Interchanging the positions of any two equations.
(ii) Multiplying an equation by a nonzero scalar.
(iii) Adding to one equation a scalar multiple of another equation.

1.4. **Gaussian elimination (L14, L15, M25).**

- (L14) Two matrices $[\mathbf{A}_1, \vec{\mathbf{b}}_1], [\mathbf{A}_2, \vec{\mathbf{b}}_2]$ of order $m \times (n+1)$ are *equivalent* if they represent equivalent systems of linear equations. Here a matrix $[\mathbf{A}, \vec{\mathbf{b}}]$ is said to *represent* a system of linear equations if it is the augmented matrix of the system. Thus solving a system can be accomplished by successively transforming its extended matrix into equivalent ones in echelon form.
- (L14) The elementary operations on systems of linear equations of the previous subsection correspond to the following *elementary row operations* on their augmented matrices:
  (E1) Interchanging any two rows.
  (E2) Multiplying any row by a nonzero scalar.
  (E3) Adding to one row of the matrix a scalar times another row of the matrix.
- (M25) Each elementary row operation can be *implemented* by an *elementary matrix* $\mathbf{E}$ in the sense that performing the operation on a matrix $\mathbf{A}$ boils down to computing the matrix product $\mathbf{EA}$.
- (L14, 15) *Gaussian elimination* is the process of transforming a matrix into row echelon form by successively applying elementary row operations.
- (L15) A *pivot* is a nonzero element of a a matrix that is used in a step of Gaussian elemination to cancel an element below it by using elementary row operation (E3).

1.5. **The inverse of a square matrix (L16, L17, L18, L19).**

- (L16) The *inverse* $\mathbf{A}^{-1}$ of a square matrix $\mathbf{A}$ is a matrix that satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.
  - If the inverse $\mathbf{A}^{-1}$ exists, is unique and satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$.
  - If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}$ is *invertible* or *non-singular.*
  - A square matrix $\mathbf{A}$ without an inverse $\mathbf{A}^{-1}$ is called *non-invertible* or *singular.*
  - A diagonal matrix $\mathbf{D}$ is invertible if, and only if, all diagonal elements are nonzero. In this case, $\mathbf{D}^{-1}$ is the diagonal matrix that has the reciprocals of the diagonal elements of $\mathbf{D}$ on the (main) diagonal.
- (L17) Let $\mathbf{A}$ be a square matrix and let $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ be a system of linear equations with coefficient matrix $\mathbf{A}$. Then:
  - When $\mathbf{A}^{-1}$ exists, then the linear system $\mathbf{A} = \vec{\mathbf{b}}$ has a unique solution that can be computed as the product $\mathbf{A}^{-1}\vec{\mathbf{b}}$.
  - When $\mathbf{A}^{-1}$ does not exist, then the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is either underdetermined or inconsistent.
- (L18) Inverse matrices can be computed by *Gauss-Jordan elimination.*
- (L19) Inverse matrices have the following properties:
  - If both $\mathbf{A}$ and $\mathbf{B}$ are invertible and of the same order, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
  - If $\mathbf{A}$ is invertible, then $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.
  - An upper- or lower-triangular matrix is invertible if, and only if, all of its diagonal elements are nonzero.
  - The inverse of an invertible upper triangular matrix is upper triangular.
  - The inverse of an invertible lower triangular matrix is lower triangular.

2. Skills

- (L12, M22) Be able to find the coefficient matrix and the extended matrix of a system of linear equations.
- (M22) Be able to find the system of linear equations that has a given extended matrix.
- (L12, M22) Be able to translate back and forth between the longhand form of a linear system and its matrix form $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$.
- (C12, M24) Recognize when a linear system whose extended matrix is in row echelon form is inconsistent. This will be the case when one of the equations takes the form $0 = 1$.
- (C12, L13, M24) Be able to find the solution set of a consistent linear system whose matrix is in row echelon form by back-substitution, both when the solution is unique and when the system is underdetermined That is, read the value of the last variable right off the last equation, substitute the value back into the second-last equation and then solve for the second last variable, substitute these two values into the third last equation, and so on, until you can solve the first equation for the first variable.

  When the system is underdetermined, its solution set can be described by leaving at least one of the variables in symbolic form, as a *free variable* or *free parameter* that can take any values. You need to choose these free variables in such a way that every choice of values for these free variables will give you a solution of the system, and all solutions can be obtained in this way.
- (L14,15; M26,27) Be able to transform a given matrix into an equivalent matrix in row echelon form by performing Gaussian elimination. Successively perform elementary row operations in such an order that you get the right type of entries for the row echelon form in successive columns, starting from the leftmost column. There is some flexibility in the order in which you work for transforming a given column. On occasion, you may need to switch some rows to get a usable pivot in the correct place.
- (L16, M28) Be able to recognize when two square matrices $\mathbf{A}$ and $\mathbf{B}$ are inverse to each other by computing $\mathbf{AB}$ and comparing this product ti the identity matrix $\mathbf{I}$ of the relevant order.
- (L17, M29) When $\mathbf{A}^{-1}$ exists and is known, be able to find the unique solution of a system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ in the form $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.
- (L18, M30) Be able to perform Gauss-Jordan elimination to determine whether $\mathbf{A}^{-1}$ exists and to find it if it does. For a given $n \times n$ matrix $\mathbf{A}$ the method works as follows:
  - Form an $n \times 2n$ matrix $\mathbf{C}$ by dropping the internal brackets in $[\mathbf{A}, \mathbf{I}_n]$ and replacing them with a vertical dividing line for visual clarity. For $n = 3$ we get:

    $$\mathbf{C} = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$
  - Perform Gaussian elimination on $\mathbf{C}$.
  - If *the first half* of the resulting matrix in echelon form has a zero row, then $\mathbf{A}$ is not invertible.
  - If the previous item does not apply, keep going and apply instances of (E3) until the first half turn into $\mathbf{I}_n$, so that the entire matrix will be in reduced echelon form. For $n = 3$, the result will look like:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right]$$

The matrix $\mathbf{B}$ in the second half will be the inverse $\mathbf{A}^{-1}$.

3. Applications (C11, M23)

An important part of Chapter was devoted to translating simple everyday word problems into systems of linear equations. While every word problem is different, basic principles of how to do such translations were illustrated in Conversation 11:

- Read the word problem closely. Keep a copy of the text handy and read it closely, several times, while working on the translation.
- Always start by setting up your variables. This is the key step.
  - Write down what each variable represents, including the units of the respective quantity.
  - It is helpful to use suggestive names for your variables.
  - Specify variables for all quantities that you might need.
- Break up the text into small pieces of information and translate one such piece at a time. You can translate them in any order, but be sure to translate all of them.
- Be sure to also translate information that is implicit in common usage of English words or phrases and that relates your variables to each other.
- Don't rush! After each step, double-check whether your translation is correct. If in doubt, you can test on a numerical example whether it makes sense.
- You want to eventually write your equations in the form $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$. But you don't need to do so right away. You can first express the info in a more convenient way, and then transform your equation into the standard format.
- At the end, double-check whether you have correctly translated all of the given information.

4. Proofs

Some proofs of properties related to inverse matrices are covered in L19, M28, and M31.