MATH 3200: OUTLINE OF CHAPTER 3
KEY CONCEPTS OF LINEAR ALGEBRA

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This chapter covers the conceptual framework of linear algebra: linear combinations, linear (in)dependence, linear span, vector spaces, rank and null space of a matrix, basis and dimension of a vector space, linear transformations. The emphasis will be on studying distinctions and connections between these concepts.

We will usually refer to specific items of the material as follows: L1 means Lecture 1, C2 means Conversation 2, and M3 means Module 3.

1. Concepts and Facts

1.1. Linear combinations and the linear span of a set of vectors (L21, L22, C21, C24).

- A vector $\vec{\mathbf{w}}$ is a *linear combination* of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ if there exist scalars $d_1, d_2, \dots, d_n$, called *coefficients*, such that $\vec{\mathbf{w}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2 + \cdots + d_n\vec{\mathbf{v}}_n$.
  - The vectors $\vec{\mathbf{w}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ are assumed to be all of the same order.
  - The zero vector $\vec{\mathbf{0}}$ of the same order as $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ is always a linear combination of these vectors.
- The set of all linear combinations of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$ is denoted by $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$ and called the *the linear span* of these vectors.
  - Determining whether a given vector is in the linear span of a given set of vectors and finding corresponding coefficients boils down to solving a system of linear equations.
  - A linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent if, and only if, $\vec{\mathbf{b}}$ is in the linear span of the column vectors of its coefficient matrix $\mathbf{A}$.
- The linear span $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$ has the following properties:
  - $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$ *is closed under multiplication by scalars,* which means that if $\vec{\mathbf{w}}$ is in this set and λ is any scalar, then $\lambda\vec{\mathbf{w}}$ is also in this set,
  - $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$ *is closed under addition of vectors,* which means that if $\vec{\mathbf{u}}, \vec{\mathbf{w}}$ are in this set, then $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ is also in this set,
  - $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$ *is closed under linear combinations,* which means that if $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \dots, \vec{\mathbf{u}}_m$ are in this set, then any linear combination $d_1\vec{\mathbf{u}}_1 + d_2\vec{\mathbf{u}} + \cdots + d_m\vec{\mathbf{u}}_m$ is also in this set.

1.2. Vector spaces (L23).

- For the purpose of this course, a *vector space* or *linear space* is a set of the form $V = span\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ for some vectors in $\mathbb{R}^n$, where k, n are some positive integers.
  - In the literature, the concept of a vector space is defined in a more general way, but in this course we will restrict ourselves only to the vector spaces that are defined in the narrower sense above. They are the most important ones for applications.
  - Each linear subspace of $\mathbb{R}^3$ is of one of the following forms:
    - A set that consists only of the origin: $\{[0, 0, 0]\} = \{\vec{\mathbf{0}}\}$.

    * A line through the origin.
    * A plane that contains the origin.
    * $\mathbb{R}^3$ itself.
  - Let V be a vector space. A set $S = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ such that $V = span(S) = span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ is called a *spanning set* of V.

1.3. **Linear dependence and linear independence of sets of vectors (C25, L24).**

- A set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ of vectors of the same order is *linearly dependent* if, and only if, there are scalars $c_1, c_2, \ldots, c_k$, not all of them zero, so that
  $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$.
  This set is *linearly independent* if, and only if, it is not linearly dependent.
  - A set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is linearly dependent if, and only if, one of these vectors can be expressed as a linear combination of the others vectors in this set. This property can be treated as an alternative definition of linear independence if $k > 1$ or if we consider the zero vector $\vec{\mathbf{0}}$ of the relevant order as a linear combination of the *empty set* $\emptyset$ of vectors.
  - A set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is linearly independent if, and only if, *every* vector $\vec{\mathbf{w}}$ in $span(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k)$ can be expressed as $\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k$ for *exactly one choice* of the coefficients $c_1, c_2, \ldots, c_k$.

1.4. **Bases and dimension of a vector space (C26, L25).**

- A linearly independent spanning set of V is called a *basis* of V.
  - Every spanning set S of a vector space V contains a basis of V.
  - Alternatively, a basis of V can also be defined as a minimal spanning set of V or as a maximal linearly independent subset of V.
  - Every two bases for the same vector space V have the same size. This size is called the *dimension* of V and denoted by $dim(V)$.
  - For a given n, we let $\vec{\mathbf{e}}_i$ denote the vector in $\mathbb{R}^n$ that has 1 in position i and 0 in all other positions. The set $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n\}$ forms the *standard basis* of $\mathbb{R}^n$.
    Its elements $\vec{\mathbf{e}}_i$ are called *standard basis vectors.*
  - If $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is a basis of V, then every vector V can be expressed as $\vec{\mathbf{w}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_n\vec{\mathbf{v}}_k$ for exactly one choice of the coefficients $c_1, c_2, \ldots, c_k$.
  - The vectors $\vec{\mathbf{c}} = [c_1, c_2, \ldots, c_k]$ of these coefficients give us *coordinates* for the elements of V and can be used to *parametrize* V. When $V = \mathbb{R}^n$ and B is the standard basis, we get the *Cartesian coordinates*; for other bases B we get *alternative coordinates* with respect to B.

1.5. **The rank, the row space, the column space, and the null space of a matrix (L26, L28).**

- The *row space* $RS(\mathbf{A}) = span(\vec{\mathbf{a}}_{1*}, \vec{\mathbf{a}}_{2*}, \ldots, \vec{\mathbf{a}}_{m*})$ of a matrix $\mathbf{A}$ is the linear span of all of its rows. The *column space* $CS(\mathbf{A}) = span(\vec{\mathbf{a}}_{*1}, \vec{\mathbf{a}}_{*2}, \ldots, \vec{\mathbf{a}}_{*n})$ of a matrix $\mathbf{A}$ is the linear span of all of its columns. $RS(\mathbf{A})$ and $CS(\mathbf{A})$ are different vector spaces, but must have the same dimension.
- The *rank* $r(\mathbf{A})$ *of* $\mathbf{A}$ is the dimension of both the column space and the row space of $\mathbf{A}$; that is, $dim(RS(\mathbf{A})) = dim(CS(\mathbf{A})) = r(\mathbf{A})$.
  - The rank $r(\mathbf{A})$ is equal to the maximum size of a linearly independent subset of its rows, aka the *row rank of* $\mathbf{A}$, and is also equal to the the maximum size of a linearly independent subset of its columns, aka as the *column rank of* $\mathbf{A}$.

  - Gaussian elimination preserves the rank of a matrix.
  - The rank of a row-reduced matrix is equal to the number of its nonzero rows, and is also equal to the number of its
  - pivotal columns, that is, columns that contain a first nonzero element of some row.
  - An $n \times n$ square matrix $\mathbf{A}$ is said to have *full rank* if $r(\mathbf{A}) = n$, that is, if its column vectors (equivalently: its row vectors) form a linearly independent set.
  - The rank $r(\mathbf{A})$ of any matrix $\mathbf{A}$ is the largest n such that $\mathbf{A}$ contains a submatrix $\mathbf{B}$ of order $n \times n$ and full rank.
- (L28, M52) The *null space* of $\mathbf{A}$ is the set of vectors $\vec{\mathbf{x}}$ such that $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
  - The null space of a matrix $\mathbf{A}$ is always a vector space.
  - The dimension of the null space is given by $dim(N(\mathbf{A})) = n - r(\mathbf{A})$.

1.6. **Connections between the solution set of a linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and the ranks of its coefficient and extended matrices $\mathbf{A}$ and $[\mathbf{A}, \vec{\mathbf{b}}]$ (L27, L29, C29).**

- The linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$is consistent if, and only if, $r(\mathbf{A}) = r([\mathbf{A}, \vec{\mathbf{b}}])$.
- When $r(\mathbf{A}) = m$, the system is always consistent; when $r(\mathbf{A}) < m$, the system is consistent for some, but not for all choices of $\vec{\mathbf{b}}$.
- When $\vec{\mathbf{x}}$ is one solution of the system, then all other solutions are vectors of the form $\vec{\mathbf{x}} + \vec{\mathbf{z}}$, where $\vec{\mathbf{z}}$ is in $N(\mathbf{A})$.
- In particular, a consistent system as above has a unique solution if, and only if, $r(\mathbf{A}) = n$ so that $dim(N(\mathbf{A})) = 0$.
- More generally, we can express the solution set of a consistent system as above by choosing $k = dim(N(\mathbf{A})) = n - r(\mathbf{A})$ among the variables as *free parameters.*

1.7. **Linear transformations (L30, C30A).**

- *Linear transformations* are important tools in linear algebra. In this course we confine ourselves to the case of functions $L : \mathbb{R}^n \to \mathbb{R}^m$, where n, m are given positive integers.
  - Such an L is called a *linear transformation* if it satisfies both of the following conditions for all vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}}$ in $\mathbb{R}^n$ and all scalars λ in $\mathbb{R}$:
    (i) $L(\lambda\vec{\mathbf{v}}) = \lambda L(\vec{\mathbf{v}})$
    (ii) $L(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = L(\vec{\mathbf{v}}) + L(\vec{\mathbf{w}})$.
  - When $\mathbf{A}$ is an $m \times n$ matrix, then the function $L_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m$ that is defined by $L(\vec{\mathbf{v}}) = \mathbf{A}\vec{\mathbf{v}}$ is a linear transformation.
  - For each linear transformation L from the space of column vectors $\mathbb{R}^n$ into the space of column vectors $\mathbb{R}^m$ there exists an $m \times n$ matrix $\mathbf{A}$ such that $L(\vec{\mathbf{x}}) = L_{\mathbf{A}}(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$. The columns of the matrix $\mathbf{A}$ for which $L = L_{\mathbf{A}}$ are the function values $L(\vec{\mathbf{e}}_i)$ of the standard basis vectors.

1.8. **How are all these concepts related to each other? (C30B).**

- The following theorems show important connections between various concepts covered in Chapters 2 and 3 for the case of a square matrix.

  Theorem 1. *Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following properties are equivalent:*
  - *$r(\mathbf{A}) = n$, that is, $\mathbf{A}$ has full rank.*
  - *The column vectors of $\mathbf{A}$ form a linearly independent set.*
  - *The row vectors of $\mathbf{A}$ form a linearly independent set.*
  - *Every $\vec{\mathbf{b}}$ in $\mathbb{R}^n$ is a linear combination of the columns of $\mathbf{A}$.*
  - *$L_{\mathbf{A}}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.*

– *Every system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is consistent.*
– *The linear transformation* $L_{\mathbf{A}}$ *is a one-to-one map.*
– *The system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *has exactly one solution.*
– *Every system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *has exactly one solution.*
– $\mathbf{A}$ *is invertible, that is,* $\mathbf{A}^{-1}$ *exists.*

Theorem 2. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Then the following properties are equivalent:*
– $r(\mathbf{A}) < n$*, that is,* $\mathbf{A}$ *does not have full rank.*
– *The column vectors of* $\mathbf{A}$ *form a linearly dependent set.*
– *The row vectors of* $\mathbf{A}$ *form a linearly dependent set.*
– *Some* $\vec{\mathbf{b}}$ *in* $\mathbb{R}^n$ *is not a linear combination of the columns of* $\mathbf{A}$*.*
– $L_{\mathbf{A}}$ *does not map* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$*.*
– *Some system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is inconsistent.*
– *The linear transformation* $L_{\mathbf{A}}$ *is not a one-to-one map.*
– *The system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *is underdetermined.*
– *Every consistent system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is underdetermined.*
– $\mathbf{A}$ *is not invertible, that is,* $\mathbf{A}^{-1}$ *does not exist.*

2. Skills

- (M42) Be able to express a given vector $\vec{\mathbf{w}}$ as a linear combination of given vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$ or determine that $\vec{\mathbf{w}}$ cannot be expressed in this form. In order to solve this type of problem, you need to:
  – Consider a linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ such that the *columns* of the coefficient matrix $\mathbf{A}$ will either be the vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$ or their transposes, and $\vec{\mathbf{b}}$ will either be $\vec{\mathbf{w}}$ or its transpose.
  – If this system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is inconsistent, then $\vec{\mathbf{w}}$ is not in $span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n)$, that is, $\vec{\mathbf{w}}$ is not a linear combination of $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$.
  – If the system is consistent, then each solution of it gives you coefficients of the linear combination.
- (M47) Be able to determine whether a given set of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ is linearly dependent or linearly independent. This boils down to:
  – Examining the solution set of a homogenous system of linear equations whose coefficient matrix has the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k$ (or their transposes, if they are row vectors) written as its columns.
  – If the system has exactly one solution, the vectors are linearly independent.
  – If the system is underdetermined, the vectors are linearly dependent.
- (C26, M48) For a given set vectors $S = \{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n\}$ be able to find a basis B for $V = span(S) = span(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n)$ that is contained in the set V. This boils down to finding a maximal linearly independent subset of S.
- (C26, M48) Determine the dimension $dim(V)$ if you are given or have found a basis B of V. This dimension will be the number of vectors in B.
- (L25B, M48) Be able to change alternative coordinates of a vector with respect to a given basis B to Cartesian coordinates and *vice versa*..
  – If $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_k\}$ is the given basis and $\vec{\mathbf{c}} = [c_1, \dots, c_k]^T$ gives the alternative coordinates of a vector $\vec{\mathbf{w}}$, then the Cartesian coordinates $\vec{\mathbf{x}}$ of this vector are given by

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{x}} \tag{1}$$

Let $\mathbf{B}$ be a matrix whose columns contain these basis vectors as columns in the given order, written in Cartesian coordinates. Then we can write (1) in matrix form as

(2) $$\vec{\mathbf{x}} = \mathbf{B}\vec{\mathbf{c}}$$

and find the Cartesian coordinates by matrix multiplication.

  - If the Cartesian coordinates $\vec{\mathbf{x}}$ are given, then we can find the alternative coordinates $\vec{\mathbf{c}}$ by solving the system of linear equations (2).

    In the special case when the vectors $\vec{\mathbf{v}}_i$ are in $\mathbb{R}^n$ and $k = n$ so that B is a basis for the entire space $\mathbb{R}^n$, then $\mathbf{B}$ is invertible and we can compute $\vec{\mathbf{c}}$ from $\vec{\mathbf{x}}$ as the matrix product

$$\vec{\mathbf{c}} = \mathbf{B}^{-1}\vec{\mathbf{x}}$$

- (L26, M49) Be able to find the rank of a given matrix $\mathbf{A}$ by performing a sufficient number of steps of a Gaussian elimination until you obtain a matrix in *generalized row echelon form.* The rank is the number of pivotal columns of this transformed matrix. It is also the number of its nonzero rows.
- (L27, M50) Be able to determine whether or not a given system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent based on comparing the ranks $r(\mathbf{A})$ and $r([\mathbf{A}, \vec{\mathbf{b}}])$. It these ranks are equal, the system is consistent; if they are not equal, the system is inconsistent.
- (L28, M52) Be able to find the null space $N(\mathbf{A})$ of a given matrix, its dimension $dim(N(\mathbf{A}))$, and to find a basis of the null space $N(\mathbf{A})$:
  - The nullspace $N(A)$ is obtained as the set of all solutions of the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
  - It's dimension is the number of free variables that are used for describing the solution set.
  - A set of basis vectors can be obtained by setting each free variable to 1 while setting all other free variables to 0.
- (L29, M54) Be able to represent the solution set of a given system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ in terms of one solution and linear combinations of a basis of the null space $N(\mathbf{A})$ of its coefficient matrix. Specifically, if $\{\vec{\mathbf{z}}_1, \ldots, \vec{\mathbf{z}}_k\}$ is a basis for the nullspace, and if $\vec{\mathbf{x}}$ is one solution of the system, then each solution can be expressed in the form $\vec{\mathbf{x}} + c_1\vec{\mathbf{z}}_1 + \ldots c_k\vec{\mathbf{z}}_k$ for some coefficients $c_1, \ldots, c_k$.
- (M55) When the values of $L(\vec{\mathbf{e}}_j)$ for a linear transformation L from the space of column vectors $\mathbb{R}^n$ into the space of column vectors $\mathbb{R}^m$ are given or can easily be inferred, be able to find an $m \times n$ matrix $\mathbf{A}$ such that $L(\vec{\mathbf{x}}) = L_{\mathbf{A}}(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$ for all $\vec{\mathbf{x}}$. The columns of this matrix must be the vectors $L(\vec{\mathbf{e}}_j)$.

3. Applications

Some applications of linear combinations and the linear span to composing meals from certain ingredients, to movement in 2 or 3 dimensions, and to chemical reaction systems were covered in C22,23,27,28 and M44,45,51. Be able to:

- (C21) Determine whether desired amounts of components in a mixture can be achieved by combining certain given ingredients, and if so, find the required amounts of the ingredients.
- (C22, M44) Determine whether a point in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be reached by moving from a given starting point in certain specified directions by a displacement that is a linear combination of these directions, and if so, find the coefficients of this linear combination.

- (C23, M45) Find the reaction vectors and the stoichiometric matrix for a given system of chemical reactions.
- (C23, M45) Determine whether a given vector of net changes can be observed in a given chemical reaction system by finding a suitable linear combination of the reaction vectors or by showing via Gaussian elimination that no such linear combination exists.

Each of the above boils down to determining the coefficients of a linear combinations or to showing that no such coefficients exist, as the case may be.

Also be able to:

- (C28, M51) Draw inferences about how many of the possible reactions in a chemical reaction system might have occurred based on an observed vector of net concentration changes and the rank of the stoichiometric matrix of this system.

4. Proofs

Some proofs of properties related to the concepts introduced in this chapter were covered in L24,27, C24,29, M43,46,52.

On the quizzes and the final you may be asked to write similar proofs.