MATH 3200: OUTLINE OF CHAPTER 4
IMPORTANT TOOLS OF LINEAR ALGEBRA

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This chapter covers determinants, eigenvectors, eigenvalues, and diagonalizations of square matrices.

We will usually refer to specific items of the material as follows: L1 means Lecture 1, C2 means Conversation 2, and M3 means Module 3.

1. Concepts and Facts

1.1. Determinants of square matrices (L31, L32).

- Every square matrix $\mathbf{A}$ has a number associated with it that is denoted

  by $\det(\mathbf{A})$ or by $\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$ and called the *determinant* of $\mathbf{A}$.
- When $\mathbf{A} = [a]$ is of order 1×1, then $\det(\mathbf{A}) = a$; for triangular matrices $\mathbf{A}$, this number $\det(\mathbf{A})$ is the product of the elements on the (main) diagonal; for all other square matrices one can calculate their determinants by either using a formula, the method of *pivotal condensation,* or the method of *cofactor expansion.* These methods are described in more detail in Section 2.

1.2. Properties of determinants (L33).

- When $\det(\mathbf{A}) = 0$ the matrix $\mathbf{A}$ is called *singular,* and when $\det(\mathbf{A}) \neq 0$, the matrix $\mathbf{A}$ is called *non-singular.*
- The formulas $\det(\mathbf{A}^T) = \det(\mathbf{A})$ and $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ always hold. The second of these formulas implies that when $\mathbf{A}$ is singular, then $\mathbf{A}^{-1}$ does not exist.
- When $\mathbf{A}$ is non-singular, then $\mathbf{A}^{-1}$ exists and $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$.
- Whether or not a square matrix is singular tells us a lot about its properties. For details see Theorems 1 and 2 of Subsection 1.9.

1.3. Minors and cofactors (L35).

- A *minor* of a matrix $\mathbf{A}$ is a determinant of any square submatrix of $\mathbf{A}$.
- The *cofactor of the element* a_{ij} of a square matrix $\mathbf{A}$ is the product of $(-1)^{i+j}$ with the minor that is obtained by removing the i^{th} row and the j^{th} column of $\mathbf{A}$.

1.4. Eigenvectors and eigenvalues (L36, C33).

- A nonzero vector $\vec{\mathbf{x}}$ is an *eigenvector* of a square matrix $\mathbf{A}$ if there exists a scalar λ such that $\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.
  - Then λ is an *eigenvalue* of $\mathbf{A}$.
  - An eigenvalue is allowed to be 0, but eigenvectors are not allowed to be zero vectors.
  - If $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, then for every scalar $c \neq 0$ the vector $c\vec{\mathbf{x}}$ is also an eigenvector of $\mathbf{A}$ with the same eigenvalue λ.

  - A square matrix $\mathbf{A}$ has an eigenvalue $\lambda = 0$ if, and only if, $\mathbf{A}$ is singular. The eigenvectors with eigenvalue 0 of $\mathbf{A}$ are the nonzero vectors in the null space $N(\mathbf{A})$.
- A square matrix $\mathbf{A}$ has a *full set of eigenvectors* if there exist n eigenvectors of $\mathbf{A}$ that form a linearly independent set.
  - For a diagonal matrix $\mathbf{D}$, the eigenvalues are the elements of the (main) diagonal, and the standard basis vectors $\vec{\mathbf{e}}_i$ form a full set of eigenvectors of $\mathbf{D}$.
  - A square matrix $\mathbf{A}$ may not have a full set of eigenvectors even if all eigenvalues are real.
  - If $\mathbf{A}$ has order $n \times n$ and has n pairwise distinct real eigenvalues, then $\mathbf{A}$ has a full set of eigenvectors.
  - A full set of eigenvectors of an $n \times n$ matrix $\mathbf{A}$ is a maximal set of eigenvectors of size n. Such a set must be a basis for $\mathbb{R}^n$. Thus $\mathbf{A}$ has a full set of eigenvectors if, and only if, there is a basis B of $\mathbb{R}^n$ that consists entirely of eigenvectors of $\mathbf{A}$.

1.5. **Geometric interpretation of eigenvectors and eigenvalues (C31, C32).**

- Let $\vec{\mathbf{x}}$ be an eigenvector of $\mathbf{A}$ with real eigenvalue λ. Then the value of the linear transformation $L_{\mathbf{A}}(\vec{\mathbf{v}})$ for each vector $\vec{\mathbf{v}}$ on the line $span(\vec{\mathbf{x}})$ is a vector $\lambda\vec{\mathbf{v}}$ on the same line $span(\vec{\mathbf{x}})$. Moreover:
  - When $|\lambda| > 1$, then along the line $span(\vec{\mathbf{x}})$ the transformation $L_{\mathbf{A}}$ will be a stretch.
  - When $|\lambda| = 1$, then $L_{\mathbf{A}}$ will preserve distances along the line $span(\vec{\mathbf{x}})$.
  - When $0 < |\lambda| < 1$, then along the line $span(\vec{\mathbf{x}})$ the transformation $L_{\mathbf{A}}$ will be a compression.
  - When $\lambda = 0$, then the line $span(\vec{\mathbf{x}})$ will be collapsed by the transformation $L_{\mathbf{A}}$ to the origin.
  - When $\lambda < 0$, then along the line $span(\vec{\mathbf{x}})$ the transformation $L_{\mathbf{A}}$ will flip directions.
- When an $n \times n$ matrix $\mathbf{A}$ with real elements has some pair of conjugate complex eigenvalues, then there exists an *eigenplane* V in $\mathbb{R}^n$ that gets mapped by $L_{\mathbf{A}}$ onto itself and is such that the restriction of $L_{\mathbf{A}}$ to V involves a rotation.

1.6. **The characteristic polynomial of a square matrix (L37A, C32, C33).**

- The *characteristic polynomial* of a square matrix $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I})$.
- The eigenvalues of $\mathbf{A}$ are then the roots of its characteristic polynomial.
- When $\mathbf{A}$ is an upper-triangular matrix or a lower-triangular matrix, then the eigenvalues of $\mathbf{A}$ are its elements on the (main) diagonal.
- For each real eigenvalue λ of $\mathbf{A}$ there exists at least one eigenvector $\vec{\mathbf{x}}$ of $\mathbf{A}$ with eigenvalue λ. The eigenvectors of $\mathbf{A}$ with eigenvalue λ are the nonzero solutions of the linear system $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$, that is, the vectors in the null space $N(\mathbf{A} - \lambda\mathbf{I})$ other than $\vec{\mathbf{0}}$.
- If λ is a root of the characteristic polynomial of a matrix $\mathbf{A}$ that is not a real number, then λ is still considered an eigenvalue of $\mathbf{A}$, but there is no eigenvector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ that is an eigenvector with eigenvalue λ.

1.7. **Eigenvalues and eigenvectors of matrix inverses and transposes (L38, C34).**

- Let $\mathbf{A}$ be an invertible matrix, and let $\vec{\mathbf{x}}$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ. Then $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{A}^{-1}$ with eigenvalue $\frac{1}{\lambda}$. In other words, $\mathbf{A}$ and $\mathbf{A}^{-1}$ have the same eigenvectors, and the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$.
- When $\mathbf{A}$ is a square matrix, then $\mathbf{A}$ and $\mathbf{A}^T$ always have the same eigenvalues, but not necessarily the same eigenvectors.

- However, when $\vec{\mathbf{x}}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, then $\vec{\mathbf{x}}^T$ is a *left eigenvector* with eigenvalue λ of $\mathbf{A}^T$, which means that $\vec{\mathbf{x}}^T\mathbf{A}^T = \lambda\vec{\mathbf{x}}^T$.

1.8. **Diagonalization and similarity of matrices (C37).**

- When B is a basis of $\mathbb{R}^n$ that consists of eigenvectors of an $n \times n$ matrix $\mathbf{A}$, then $\mathbf{A} = \mathbf{BDB}^{-1}$, where $\mathbf{D}$ is the *diagonalization* of $\mathbf{A}$ that lists the respective eigenvalues of the vectors in B on the (main) diagonal and has zero elements in all off-diagonal places.
- A square matrix $\mathbf{A}$ is *diagonalizable* if, and only if, $\mathbf{A}$ is *similar* to a diagonal matrix $\mathbf{D}$, which means that $\mathbf{D} = \mathbf{B}^{-1}\mathbf{AB}$ for some invertible matrix $\mathbf{B}$.
- A square matrix $\mathbf{A}$ is diagonalizable if, and only if, it has a full set of eigenvectors.

1.9. **Important connections between concepts introduced in Chapters 3 and 4.**

Theorem 1. *Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following properties are equivalent:*

- $\det(\mathbf{A}) \neq 0$*, that is,* $\mathbf{A}$ *is non-singluar.*
- $\lambda = 0$ *is not an eigenvalue of* $\mathbf{A}$*.*
- $\mathbf{A}$ *is invertible, that is,* $\mathbf{A}^{-1}$ *exists.*
- $r(\mathbf{A}) = n$*, that is,* $\mathbf{A}$ *has full rank.*
- *The column vectors of* $\mathbf{A}$ *form a linearly independent set.*
- *The row vectors of* $\mathbf{A}$ *form a linearly independent set.*
- *Every* $\vec{\mathbf{b}}$ *in* $\mathbb{R}^n$ *is a linear combination of the columns of* $\mathbf{A}$*.*
- *The linear transformation* $L_{\mathbf{A}}$ *maps* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$*.*
- *Every system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is consistent.*
- *The linear transformation* $L_{\mathbf{A}}$ *is a one-to-one map.*
- *The system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *has exactly one solution.*
- *Every system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *has exactly one solution.*

Theorem 2. *Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following properties are equivalent:*

- $\det(\mathbf{A}) = 0$*, that is,* $\mathbf{A}$ *is singluar.*
- $\lambda = 0$ *is an eigenvalue of* $\mathbf{A}$*.*
- $\mathbf{A}$ *is not invertible, that is,* $\mathbf{A}^{-1}$ *does not exist.*
- $r(\mathbf{A}) < n$*, that is,* $\mathbf{A}$ *does not have full rank.*
- *The column vectors of* $\mathbf{A}$ *form a linearly dependent set.*
- *The row vectors of* $\mathbf{A}$ *form a linearly dependent set.*
- *Some* $\vec{\mathbf{b}}$ *in* $\mathbb{R}^n$ *is not a linear combination of the columns of* $\mathbf{A}$*.*
- *The linear transformation* $L_{\mathbf{A}}$ *does not map* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$*.*
- *Some system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is inconsistent.*
- *The linear transformation* $L_{\mathbf{A}}$ *is not a one-to-one map.*
- *The system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *is underdetermined.*
- *Every consistent system* $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *is underdetermined.*

2. Skills

You will learn how to:

- **Calculate the determinant of a** 2×2 **matrix** from the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- **Calculate determinants with the method of pivotal condensation (L32, M62).**
  (1) Transform $\mathbf{A}$ by successive elementary row operations into
  $\mathbf{A} \to \mathbf{E}_1\mathbf{A} \to \mathbf{E}_2\mathbf{E}_1\mathbf{A} \to \cdots \to \mathbf{E}_k \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U}$, where $\mathbf{U}$ is upper-triangular.
  (2) Keep track of how the determinant changes at every step of applying an elementary row operation:
    - When we switch two rows, the determinant switches sign.
    - When we multiply a row of the matrix by a scalar λ, the determinant also changes by a factor of λ.
    - When we add a scalar multiple of one row to another row, the determinant remains unchanged.

  (3) Calculate $\det(\mathbf{U})$ as the product of the diagonal elements.
  (4) Deduce $\det(\mathbf{A})$ from $\det(\mathbf{U})$ and your record on how the determinant did or did not change at every step.
- **Calculate** $\det(\mathbf{A})$ **by cofactor expansion (L35, M65).**
  (1) Pick any row or column.
  (2) For each element of the chosen row or column, find its cofactor.
  (3) Multiply each element in the chosen row or column by its cofactor.
  (4) Sum the results.

  Cofactor expansion works best if you choose the row or column along which you expand as one that contains many zero elements.
- **Determine whether a given vector $\vec{\mathbf{x}}$ is an eigenvector of a given square matrix A and if it is, to find its eigenvalue (L36, M66).**
  - Inspect the product $\mathbf{A}\vec{\mathbf{x}}$ and check whether it is equal to $\lambda\vec{\mathbf{x}}$ for some scalar λ.
- **Find all eigenvalues and a maximal linearly independent set of eigenvectors of a given square matrix A (L37AB, M67AB).**
  (1) Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I})$.
  (2) Factor the characteristic polynomial. The roots are the eigenvalues of $\mathbf{A}$.
  (3) For each (real) eigenvalue λ_i, find the eigenvectors $\vec{\mathbf{x}}$ with this eigenvalue as follows:
    (a) Form $\mathbf{A} - \lambda_i\mathbf{I}$ by subtracting the number λ_i from each diagonal element of $\mathbf{A}$.
    (b) Solve the system of linear equations $(\mathbf{A} - \lambda_i\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$, for example by Gaussian elimination.
    (c) Your solution will contain *at least 1* and *up to* k_i variables x_j that you can choose arbitrarily. Here k_i denotes the multiplicity of eigenvalue λ_i. For each of these variables x_j, find an eigenvector by setting it to 1, while setting the other variables that you can choose freely to 0.
- **Find a square matrix A that has a specified full set of eigenvectors with specified eigenvalues (C37, M72).**
  - Calculate $\mathbf{A} = \mathbf{B}\mathbf{D}\mathbf{B}^{-1}$, where columns of $\mathbf{B}$ are the specified eigenvectors and $\mathbf{D}$ is the diagonal matrix that lists the respective eigenvalues of these eigenvectors on the main diagonal in the same order.

3. Applications

3.1. Determinants and properties of linear transformations (L34, M64).

- The determinant of a 2×2 or 3×3 matrix $\mathbf{A}$ tells us whether and how the corresponding linear transformation $L_{\mathbf{A}}$ changes orientation and area or volume.
  - When $\det(\mathbf{A}) > 1$, the transformation $L_{\mathbf{A}}$ preserves orientation; when $\det(\mathbf{A}) < 1$, the transformation $L_{\mathbf{A}}$ reverses orientation.
  - Assume $\mathbf{A}$ has order 2×2. Then $L_{\mathbf{A}}$ will change areas by a factor of $|\det(\mathbf{A})|$. In particular, if $|\det(\mathbf{A})| = 1$, then $L_{\mathbf{A}}$ will preserve areas.
  - Assume $\mathbf{A}$ has order 3×3. Then $L_{\mathbf{A}}$ will change volumes by a factor of $|\det(\mathbf{A})|$. In particular, if $|\det(\mathbf{A})| = 1$, then $L_{\mathbf{A}}$ will preserve volumes.

3.2. Applications of eigenvectors and eigenvalues.

- Diagonalization of square matrices with full sets of eigenvalues. See Subsection 1.8
- Stationary distributions of a Markov chain are left eigenvectors with eigenvalue $\lambda = 1$. See C36 and M71.

4. Proofs

A number of proofs were covered in Chapter 4. Most notably:

- Proofs of several properties of determinants of 2×2 and 3×3 matrices in L31, M61, and M63B.
- Proofs of some properties of eigenvectors and eigenvalues in L36, C34, M66, M67A, M68A, and M72.

On the quizzes and the final you may be asked to write similar proofs.