MATH 3200: OUTLINE OF CHAPTER 5
IMPORTANT TOOLS OF LINEAR ALGEBRA

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Covers vector norms, inner products, orthogonality, orthonormal bases, and least squares solutions.

We will usually refer to specific items of the material as follows: L1 means Lecture 1, C2 means Conversation 2, and M3 means Module 3.

1. Concepts and Facts

1.1. Norms and distances (L41, M81).

- The *norm* of a vector $\vec{\mathbf{x}}$ is a real number $\|\vec{\mathbf{x}}\|$ that can be thought of as its length. It has the following properties:
  (i) $\|\vec{\mathbf{x}}\| \geq 0$ and $\|\vec{\mathbf{x}}\| = 0$ if, and only if, $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
  (ii) If α is any scalar, then $\|\alpha\vec{\mathbf{x}}\| = |\alpha|\,\|\vec{\mathbf{x}}\|$.
  (iii) $\|\vec{\mathbf{x}} + \vec{\mathbf{y}}\| \leq \|\vec{\mathbf{x}}\| + \|\vec{\mathbf{y}}\|$ for any vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$.
- There are many different norms on a given vector space. As in the lectures, in this review we focus on the familiar *Euclidean norm* $\|\cdot\|$ that is defined as follows:
  $\|\vec{\mathbf{x}}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.
- A *unit vector* is a vector $\vec{\mathbf{x}}$ such that $\|\vec{\mathbf{x}}\| = 1$.
- The *normalization* of $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ is the vector $\frac{\vec{\mathbf{x}}}{\|\vec{\mathbf{x}}\|}$.
- The *distance* $d(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ between vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ is defined as $d(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = \|\vec{\mathbf{x}} - \vec{\mathbf{y}}\|$.

1.2. Orthogonality (L42, M82).

- The *(standard) inner product* $\langle\vec{\mathbf{x}}, \vec{\mathbf{y}}\rangle$ aka *dot product* of vectors $\vec{\mathbf{x}} = [x_1, x_2, \ldots, x_n]$ and $\vec{\mathbf{y}} = [y_1, y_2, \ldots, y_n]$ is defined as
  $\langle\vec{\mathbf{x}}, \vec{\mathbf{y}}\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{i=1}^n x_iy_i$.
- $\sqrt{\langle\vec{\mathbf{x}}, \vec{\mathbf{x}}\rangle} = \|\vec{\mathbf{x}}\|$ is the Euclidean norm of $\vec{\mathbf{x}}$. Thus $\langle\vec{\mathbf{x}}, \vec{\mathbf{x}}\rangle = \|\vec{\mathbf{x}}\|^2$.
- Two vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ are *orthogonal* if, and only if, $\langle\vec{\mathbf{x}}, \vec{\mathbf{y}}\rangle = 0$.
- The angle Θ in $[0, \pi]$ between two nonzero vectors in $\mathbb{R}^n$ can be computed from the Law of Cosines

$$\cos\Theta = \frac{\langle\vec{\mathbf{x}}, \vec{\mathbf{y}}\rangle}{\|\vec{\mathbf{x}}\|\,\|\vec{\mathbf{y}}\|}.$$

1.3. Orthogonal projections, orthogonal complements, and orthonormal bases (L43, M83).

- Let $\vec{\mathbf{a}}, \vec{\mathbf{x}}$ be two vectors in $\mathbb{R}^n$. Then there exists exactly one pair $(\vec{\mathbf{u}}, \vec{\mathbf{v}})$ of vectors in $\mathbb{R}^n$ such that
  - $\vec{\mathbf{x}} = \vec{\mathbf{u}} + \vec{\mathbf{v}}$.
  - $\vec{\mathbf{u}} = c\vec{\mathbf{a}}$ for some scalar c, that is, $\vec{\mathbf{u}}, \vec{\mathbf{a}}$ are on the same line.
  - $\langle\vec{\mathbf{a}}, \vec{\mathbf{v}}\rangle = \langle\vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle$, that is, $\vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{a}}$ and $\vec{\mathbf{u}}$.

The vector $\vec{\mathbf{u}}$ is called the *projection of* $\vec{\mathbf{x}}$ *onto* $\vec{\mathbf{a}}$,
and $\vec{\mathbf{v}}$ is called the *orthogonal complement of* $\vec{\mathbf{x}}$ *with respect to* $\vec{\mathbf{a}}$.

When $\vec{\mathbf{a}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ and $\vec{\mathbf{v}} = \vec{\mathbf{x}}$.

When $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$, then

$$\vec{\mathbf{u}} = \frac{\langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{a}}, \vec{\mathbf{a}} \rangle} \vec{\mathbf{a}} \qquad \text{and} \qquad \vec{\mathbf{v}} = \vec{\mathbf{x}} - \vec{\mathbf{u}} = \vec{\mathbf{x}} - \frac{\langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle}{\langle \vec{\mathbf{a}}, \vec{\mathbf{a}} \rangle} \vec{\mathbf{a}}.$$

- A set $B = \{\vec{\mathbf{b}}_1, \dots, \vec{\mathbf{b}}_n\}$ of vectors is *orthogonal* if $\langle \vec{\mathbf{b}}_i, \vec{\mathbf{b}}_j \rangle = 0$ for all $i \neq j$.
- We call B *orthonormal* if it is orthogonal and composed of unit vectors, so that $\|\vec{\mathbf{b}}_i\| = \langle \vec{\mathbf{b}}_i, \vec{\mathbf{b}}_i \rangle = 1$ for all $i = 1, \dots, n$.
- A set B is an *orthonormal basis* for a linear subspace V of $\mathbb{R}^n$ if it is orthonormal and a basis for V.
- $B = \{\vec{\mathbf{e}}_1, \dots, \vec{\mathbf{e}}_n\}$ is an example of an orthonormal basis for $\mathbb{R}^n$.
- Every vector space V has an orthonormal basis B. Such a basis can be found by performing *Gram-Schmidt orthonormalization* on a given spanning set A for V.

2. Skills (M81, M82, M83)

- Be able to compute the Euclidean norm $\|\vec{\mathbf{x}}\|$ of a given vector $\vec{\mathbf{x}}$ and determine whether $\vec{\mathbf{x}}$ is a unit vector.
- Be able to normalize a given vector $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ with respect to the Euclidean norm.
- Be able to compute the Euclidean distance between two given vectors.
- Be able to compute the inner product of two given vectors.
- Be able to determine whether two given vectors are orthogonal.
- Be able to find the angle between two given vectors using the Law of Cosines.
- For any given vectors $\vec{\mathbf{x}}, \vec{\mathbf{a}}$, be able to compute the projection of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{a}}$ and the orthogonal complement of $\vec{\mathbf{x}}$ with respect to $\vec{\mathbf{a}}$.
- For a given set B of vectors of the same order, be able to determine whether this set is orthogonal and/or orthonormal.

3. Applications (L43, M83, C41)

- When B is an orthonormal basis for a vector space V, then every vector $\vec{\mathbf{x}}$ can be written in the alternative coordinates $\vec{\mathbf{c}}$ with respect to B that are given by

  $\vec{\mathbf{c}} = [\langle \vec{\mathbf{b}}_1, \vec{\mathbf{x}} \rangle, \langle \vec{\mathbf{b}}_2, \vec{\mathbf{x}} \rangle, \dots, \langle \vec{\mathbf{b}}_n, \vec{\mathbf{x}} \rangle]$.

  Be able to use this formula for finding the alternative coordinates c_i wrt B for a given vector $\vec{\mathbf{x}}$.
- Orthonormal bases can be used to compute *least-squares solutions* $\vec{\mathbf{x}}$ of systems of linear equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ that are overconstrained. While not actually a solution, a least-squares solution $\vec{\mathbf{x}}$ has the property that the Euclidean distance of $\mathbf{A}\vec{\mathbf{x}}$ from $\vec{\mathbf{b}}$ is as small as possible. Such $\vec{\mathbf{x}}$ can be computed by first finding the the orthogonal projection $\vec{\mathbf{u}}$ of $\vec{\mathbf{b}}$ onto the linear span of the columns of $\mathbf{A}$ as the sum of the orthogonal projections of $\vec{\mathbf{b}}$ onto the vectors of the orthonormal basis, and then finding $\vec{\mathbf{x}}$ as the solution of the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$. *While this procedure was briefly illustrated in Conversation 41, it will not be a topic for the final.*

4. Proofs

A proof of basic properties of the standard inner product is covered in M82B.
On the final you may be asked to write similar proofs.