Conversation 1: Hi!

Winfried Just
Department of Mathematics, Ohio University

Companion to Advanced Calculus
Hi! Meet Alice, Bob, and Cindy

Alice: Hi! I’m Alice, and I’m really looking forward to this class.

Bob: Hi! I’m Bob. Nice to meet you, Alice!

Alice: Nice to meet you, Bob!

Bob: I’m also looking forward to this course. They say it will be “different”. I wonder: What is it going to be like?

Cindy: Hi! I’m Cindy. I wonder about this too. Sounds a bit scary.

Alice and Bob: Hi Cindy! Nice to meet you!

Cindy: Nice to meet you, Alice and Bob!
Do you think this class will be difficult?

Alice: I think it’s going to be challenging. I like challenge.

Bob: More difficult than other classes, I heard. But students of this prof who study hard tend to do well, so we should be fine.
Denny: Don’t count on that! A friend of mine took a class with this prof, pulled all-nighters before before each test and the final, and still barely got a D.

Bob: I was talking about studying regularly.

Denny: Btw, I’m Denny. Hi everybody!

The others: Hi Denny! Nice to meet you!

Frank: Hi! I’m Frank. Nice to meet you all.

The others: Hi Frank! Nice to meet you!

Frank: The class will be insanely hard, I can tell you. This prof is unreasonable.

The others: What do you mean?

Frank: I heard he wants to focus on theory and proofs all the time instead of teaching us how to calculate stuff.
Bob: Well, that’s exactly what the course description says: ”A proof-based course ... ”

Frank: That’s unreasonable. As a physics major, I need math for doing calculations, not for proving theorems.

Alice: In physics, don’t you need to clearly distinguish between a valid reasoning and a faulty one?

Frank: We damn sure need to do this. All the time.

Alice: Proofs are in essence tools for making this distinction. Both in pure and in applied mathematics.

Cindy: But proofs are so scary!! I never know how to do them!!!

Alice: You will learn in this class how to do them. Constructing proofs is in large part a craft. Like any other craft, it can be learned by practising it. I think in this course we will gain a lot of practice with writing proofs and evaluating their correctness.
Bob: Sounds like at the end of this course we will all be able to do proofs. Nothing to be scared of, Cindy!

It reads here further in the course description: “Topics include properties of the real and complex numbers, metric spaces and basic topology, sequences and series, a careful study of limits and continuity, differentiation and Riemann-Stieltjes integration.”

Denny: I thought this was a course in Advanced Calculus. What does all this stuff have to do with calculus?

Theo: These are concepts on which the computational procedures of calculus are based. Advanced calculus, also known as analysis, rigorously studies properties of these and related concepts.

Denny: If you say so ... But won’t you at least rigorously introduce yourself, if I may use your expression?

Theo: Hi everybody! I’m Theo. Delighted to meet you all.

The others: Hi Theo! Nice to meet you!
Frank: That’s exactly what I was afraid of: We will be taught a lot of mathematical theory instead of how to perform some useful calculations.

Theo: The conceptual understanding gained by studying the mathematical constructs listed in the course description is needed for correctly performing advanced calculations.

Bob: How would that be? To obtain the correct result, we need to strictly follow the right recipe, not study abstract concepts.

Alice: Consider, for example, the following limit of integrals:

$$\lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{1}{1 + (x - y)^2} \, dx$$

How would you calculate this integral, Bob?

Bob: Hmm. I don’t recall offhand the procedure ...

Alice: Perhaps we can figure it out together.
Conceptual understanding vs. calculations: An example

\[
\lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{1}{1 + (x - y)^2} \, dx
\]

**Denny:** I don’t recall it either, but we can simply swap the integral sign and the limit and then the problem becomes easier:

\[
\int_{-\infty}^{\infty} \lim_{y \to \infty} \frac{1}{1 + (x - y)^2} \, dx = \int_{-\infty}^{\infty} 0 \, dx = 0.
\]

**Cindy:** Can you explain how you got that 0 in the integrand, please?

**Denny:** That’s because \( x \) is fixed here and \((x - y)^2\) gets larger and larger.

**Cindy:** I still don’t understand it.
**Bob:** Denny was using the limit laws that we learned in Calc I:

\[
\lim_{y \to \infty} \frac{1}{1 + (x - y)^2} = \frac{1}{\lim_{y \to \infty} (1 + (x - y)^2)}
\]

\[
= \frac{1}{\lim_{y \to \infty} 1 + \lim_{y \to \infty} (x - y)^2} = \frac{1}{1 + (\lim_{y \to \infty} (x - y))^2}
\]

\[
= \frac{1}{1 + (\lim_{y \to \infty} x - \lim_{y \to \infty} y)^2} = \frac{1}{1 + (x - \infty)^2}
\]

Since \(x\) is fixed, the denominator increases without bound, and the limit must be 0.

**Cindy:** Thank you, Bob, for explaining this so nicely! Now I can see that Denny’s calculation of the limit is correct.
(Why) do we need limit laws?

**Denny:** Except that I didn’t use no limit laws or any such theoretical stuff. I saw right off the bat that the limit was zero.

**Alice:** But in a more complicated example, if you had not seen right off the bat what the limit was, what would you have done?

**Denny:** Dunno.

**Bob:** Then you might still have been able to use limit laws and work out the answer step by step.

**Denny:** Yeah ... perhaps.

**Frank:** But wait! We don’t need any limit at all. The integral

\[
\int_{-\infty}^{\infty} \frac{1}{1 + (x - y)^2} \, dx
\]

does not depend on \( y \).

**The others:** Why not?
Another approach to $\lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} \, dx$

**Frank:** Make the substitution $u = x - y$.
Then $\frac{du}{dx} = 1$ so that $du = dx$.
Also, for fixed $y$, when $x \to \infty$, then $u \to \infty$, and when $x \to -\infty$, then $u \to -\infty$.
So we get for all $y$:

$$\int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} \, dx = \int_{-\infty}^{\infty} \frac{1}{1+u^2} \, du.$$

**Bob:** And then by the limit law for a constant function

$$\lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} \, dx = \lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{1}{1+u^2} \, du = \int_{-\infty}^{\infty} \frac{1}{1+u^2} \, du.$$

**Denny:** So by my calculations, the integral on the right must be 0.

**Frank:** But it isn’t.

**Denny:** What is it then?
Cindy: In calculus we learned that
\[ \int \frac{1}{1 + u^2} \, du = \arctan u + C. \]

So for the improper integral we get
\[ \int_{-\infty}^{\infty} \frac{1}{1 + u^2} \, du = \arctan u \bigg|_{u=-\infty}^{u=\infty} \]

This means that
\[ \int_{-\infty}^{\infty} \frac{1}{1 + u^2} \, du = \lim_{u \to \infty} \arctan u - \lim_{u \to -\infty} \arctan u = \frac{\pi}{2} - \frac{-\pi}{2} = \pi. \]

Alice and Bob: Very well done, Cindy!

Cindy: Thank you!
I can do calculations, but I never can do proofs!

Alice: By the end of this course sequence, you will be able to do proofs as well as you now do calculations.
Denny: But why did Cindy get $\pi$ while I got 0?

Bob: Good question. Let’s review our calculations. Cindy showed that

$$\lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{1}{1 + (x - y)^2} \, dx = \pi.$$ 

Denny showed that

$$\int_{-\infty}^{\infty} \lim_{y \to \infty} \frac{1}{1 + (x - y)^2} \, dx = 0.$$ 

Frank: Means you cannot swap the integral sign and the limit.

Denny: But it should give the same result! I have seen other people doing this, and it worked just fine!
Alice: In fact, it is often the case that for a function $f(x, y)$ of two variables we have

$$\lim_{y \to \infty} \int_{-\infty}^{\infty} f(x, y) \, dx = \int_{-\infty}^{\infty} \lim_{y \to \infty} f(x, y) \, dx.$$ 

Often, but not always, as we have seen in this example.

Frank: So how can we know when the two numbers are equal?

Denny: And simplify a calculation by swapping, as I did?

Theo: This is exactly where the study of analysis helps. This subject provides us with theorems that give precise conditions on the mathematical objects involved, in this case on the function $f(x, y)$, where a certain method of calculation, like Denny’s swapping of the integral sign and the limit will lead to correct results.

Bob: I see. Like in the limit laws we used earlier today.
How are we going to learn all this?

**Denny:** Are you saying, Theo, that I was sort of right with my swapping the limit and the integral sign trick?

**Theo:** But you got the wrong result. In this course sequence, you will learn how to get from “sort of right” to being confident that a given method is correct.

**Frank:** Sounds somewhat plausible. But as I said: This course will be mostly about theory and will be insanely hard.

**Alice:** Advanced calculus is a very challenging subject, yes. But we can all learn to understand it really well.

**Cindy:** It still all sounds so scary. How can we possibly learn all this theory?

**Alice:** By talking about it, for starters. Like the six of us are talking right now about the class and the subject. Bouncing around some ideas, getting it wrong much of the time, sometimes right, and giving each other feedback.
What’s this course going to be like?

Cindy: You mean ... like in group work?

Alice: Yeah. But I heard that this prof has an idea for doing something like group work in front of the whole class. In fact, together with the whole class.

Frank: Crazy.

Cindy: I like crazy ideas. Well, some really cool ones anyway.

Denny: As long as it will help me with my grade ...

Bob: How on earth is this going to work?

Alice: Look at all these people sitting there. Shall we meet them?

Question C1.1: Hi! Who are you?

All: (Whisper) Who is the guy listening in on our conversation? (Aloud) Hi! Who are you?

WJ: Hi! I’m Winfried Just, your professor for this class. And I will now go with you over the syllabus.
**Take-home message**

*Advanced calculus* aka *analysis* studies properties of the mathematical objects that are used in the computational procedures of calculus. Its goal is to derive rigorous theorems about these objects that can be used to construct correct computations.

This is a proof-based course sequence and an important goal is to teach you how to construct proofs and evaluate the correctness of proofs and mathematical arguments in general. This is in large part a craft and can be learned by practising it. Your key to success is staying current with the material, doing all the exercises, and actively participating in the class sessions.

Part of the course will be taught with Top Hat’s interactive response system. A number of presentations will feature the six fictional characters introduced here. None of them is based on any one real person. They represent different cognitive styles and attitudes that will contribute in various ways to success with the mathematical problems that they discuss.