Conversation 24: Introduction to power series

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Companion to Advanced Calculus

The definition of power series

Cindy: I heard that the material on convergence of sequences and series of functions is mainly needed for studying power series, which are very important in analysis.

What are these power series?

Bob: The textbook gives the following definition:

Definition 4.1.1: (Formal power series) Let a be a real number. A formal power series centered at a is any series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$, where $c_0, c_1, \dots$ is a sequence of real numbers (not depending on x).
We refer to c_n as the n^{th} coefficient of this series.

Note that each term $c_n(x-a)^n$ in this series is a function of a real variable x.

Cindy: Like, $\sum_{n=0}^{\infty} \frac{1}{n+1}(x+5)^n$ and $\sum_{n=0}^{\infty} 3^x(x-1)^n$.
These would be formal power series, right?

Question C24.1: Did Cindy get this right?

Alice: $\sum_{n=0}^{\infty} \frac{1}{n+1}(x+5)^n$ would be, since here $a = -5$ and $c_n = \frac{1}{n+1}$ is a real number for each $n \in \mathbb{N}$. But $\sum_{n=0}^{\infty} 3^x(x-1)^n$ would not be, since 3^x is a function of x, not a fixed real number.

Denny: Why does this definition say "formal"?

Theo: Because it does not make any mention of whether or not the series converge for some x.

The series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is only defined as an expression.

Denny: So how about that convergence?

Theo: For every formal power series $\sum_{n=0} c_n(x-a)^n$ there exists an extended real number $R \in [0, +\infty]$, called the radius of convergence of the series, such that:

- for every r with $0 \leq r < R$ the series converge uniformly on the interval $[-r+a, a+r]$,
- and for every $x \notin [-R+a, a+R]$ the series diverges.

Radius of convergence of a power series

Bob: I can see here that the textbook defines this radius of convergence as $R := \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}$.

Frank: Horrible! All of this looks way too complicated. Let's simplify things and focus on the special case when $a = 0$. Then the power series becomes $\sum_{n=0}^{\infty} c_n x^n$.

Cindy: This looks much more friendly, yes. But I am worried that if we prove things for this special case, would we then not need to redo all the proofs for the general case?

Alice: Good question, Cindy! But Frank was suggesting a very useful strategy for studying new concepts: Simplify your notation as much as possible.

When we want to study behaviour of a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ on an interval $[-r+a, a+r]$, we can introduce a new variable $y := x - a$, so that $x \in [-r+a, a+r]$ if, and only if, $y \in [-r, r]$, and study convergence of $\sum_{n=0}^{\infty} c_n y^n$ on the interval $[-r, r]$ instead.

Frank: You are right, Alice. Let's only talk about power series $\sum_{n=0}^{\infty} c_n x^n$ today and leave the translations into the general case to our prof for the next lecture.

Denny: Yeah. Let's keep it as simple as possible.
So when is $\sum_{n=0}^{\infty} c_n x^n$ convergent?

Theo: It is usually easier to determine for which x the series $\sum_{n=0}^{\infty} c_n x^n$ is absolutely convergent.

Denny: How would that be easier?

Theo: Recall the root test for a series $\sum_{n=0}^{\infty} a_n$ of real numbers.
It says that when $\limsup_{n\to\infty} |a_n|^{1/n} < 1$,
then $\sum_{n=0}^{\infty} a_n$ is absolutely convergent,
and when $\limsup_{n\to\infty} |a_n|^{1/n} > 1$, then $\sum_{n=0}^{\infty} a_n$ is divergent.

Cindy: But here we don't have $\sum_{n=0}^{\infty} a_n$ for fixed numbers a_n.
It's more like, we have infinitely many such series, one for each x.

Alice: Excellent observation, Cindy!

Cindy: Do you mean, like, if we fix x and let $a_n := c_n x^n$ for all $n \in \mathbb{N}$, then $\sum_{n=0}^{\infty} c_n x^n$ will be absolutely convergent when $\limsup_{n\to\infty} |c_n x^n|^{1/n} < 1$,
and will be divergent when $\limsup_{n\to\infty} |c_n x^n|^{1/n} > 1$?

Alice: Yes, this is what I thought you had in mind.

Cindy: So now we can write this as $\limsup_{n\to\infty} |c_n x^n|^{1/n} = \limsup_{n\to\infty} |c_n|^{1/n} |x^n|^{1/n} = |x| \limsup_{n\to\infty} |c_n|^{1/n}$, right?

Question C24.2: Did Cindy get this right?

Bob: Excellent observation, Cindy!
So assume $0 < \limsup_{n\to\infty} |c_n|^{1/n} < \infty$.
Then $|x| \limsup_{n\to\infty} |c_n|^{1/n} < 1$, iff $|x| < \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}$.
The series $\sum_{n=0}^{\infty} c_n x^n$ will be absolutely convergent in this case.

More about the radius of convergence

Similarly, $|x| \limsup_{n\to\infty} |c_n|^{1/n} > 1$ iff $|x| > \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}$.

The series $\sum_{n=0}^{\infty} c_n x^n$ will be divergent in this case.

Notice that $\frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}$ is the radius of convergence R as specified in Definition 4.1.3.

Frank: Now I see. So $\sum_{n=0}^{\infty} c_n x^n$ is absolutely convergent for all $x \in (-R, R)$, and is divergent when $|x| > R$.

Theo: Exactly! We have essentially proved parts (a) and (b) of Theorem 4.1.6.

Cindy: But wait! What if $\limsup_{n\to\infty} |c_n|^{1/n} = 0$?

Bob: Then we let $R := +\infty$ and conclude that $\sum_{n=0}^{\infty} c_n x^n$ is absolutely convergent for all $x \in (-\infty, +\infty) = (-R, R) = \mathbb{R}$.

Cindy: And what if $\limsup_{n\to\infty} |c_n|^{1/n} = \infty$?

Bob: Then we let $R := 0$.

What happens when $|x| = R$?

Cindy: So $\sum_{n=0}^{\infty} c_n x^n$ would then be divergent for all $x \in \mathbb{R}$?

Question 24.3: Did Cindy get this right?

Frank: No. $\sum_{n=0}^{\infty} c_n x^n$ would then be divergent only for all x with $x \neq 0$, that is, all x with $|x| > R = 0$.
But $\sum_{n=0}^{\infty} c_n 0^n = c_0 + 0 + 0 + \dots$ would still be convergent.

Denny: Interesting. What happens if $x = R$ or $x = -R$?
The proof we have found doesn't tell us.

Theo: This is quite complicated. For $x \in \{-R, R\}$ the series may be absolutely convergent, conditionally convergent, or divergent.
It all depends on the particular choices of the coefficient c_n.

Bob: We will see examples in Module 57;
so let's not discuss this right now.

Theo, you had mentioned that when $0 \leq r < R$, then the series $\sum_{n=0}^{\infty} c_n x^n$ is uniformly convergent on $[-r, r]$.
Can you tell us more about this?

Uniform convergence of a power series

Theo: With pleasure. Now we need to think about $\sum_{n=0}^{\infty} c_n x^n$ as a series of power functions $c_n x^n : [-r, r] \to \infty$.
Note that for all $x \in [-r, r]$ and all $n \in \mathbb{N}$ we have $|c_n x^n| \leq |c_n r^n|$.

Cindy: Would this be, like, saying that $||c_n x^n||_\infty \leq |c_n r^n|$? So that we could use the Weierstrass M-test if $\sum_{n=0}^{\infty} |c_n r^n|$ is convergent?

Question C24.4: Did Cindy get this right?

Theo: Exactly! And when $0 \leq r < R$, then we are guaranteed that $\sum_{n=0}^{\infty} c_n r^n$ is absolutely convergent. So $\sum_{n=0}^{\infty} c_n x^n$ must be uniformly convergent on $[-r, r]$, which is Theorem 4.1.6(c).

Denny: Could you also say that $\sum_{n=0}^{\infty} c_n x^n$ is uniformly convergent on $(-R, R)$?

Theo: You can say it of course, but it would not always be true. Sometimes it is true, sometimes not. It all depends on the choices of the coefficients c_n.

Continuity on the interval of convergence

Bob: Let me see whether I got all of this straight. A power series $\sum_{n=0}^{\infty} c_n x^n$ defines a function $f(x)$ on an interval I that is centered around 0, such that $\sum_{n=0}^{\infty} c_n x^n$ converges to $f(x)$ for all $x \in I$ and diverges for all $x \notin I$.

Theo: Exactly! The interval I is called the interval of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$.
It could be of the form $(-R, R)$, $[-R, R)$,$(-R, R]$ or $[-R, R]$, where R is the radius of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$.

Bob: Now for every $0 \leq r < R$ the power series $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly to $f \restriction [-r, r]$. So, since each function $c_n x^n$ is continuous, the restriction $f \restriction [-r, r]$ must also be a continuous function.

Denny: And since every $x_0 \in (-R, R)$ is in some interval $[-r, r]$ for some $r < R$, the function $f \restriction (-R, R)$ must also be continuous!

Theo: Exactly!

Integrating power series term by term

Cindy: So if $f(x) = \sum_{n=0}^{\infty} c_n x^n$ on $(-R, R)$ and $[y, z] \subseteq (-R, R)$ is a closed and bounded sub-interval, could we then also calculate $\int_{[y,z]} f = \sum_{n=0}^{\infty} \int_{[y,z]} c_n x^n$, I mean, integrate term by term?

Bob: Yes, Cindy! For $[y, z] \subseteq (-R, R)$, we always can find $r < R$ such that $[y, z] \subseteq [-r, r]$.
Then the partial sums $f^{(N)}$ defined by $f^{(N)}(x) := \sum_{n=0}^{N} c_n x^n$ converge uniformly to f on $[y, z]$, and by Theorem 3.6.1 the function $f \restriction [y, z]$ is Riemann integrable and $\int_{[y,z]} f = \lim_{N\to\infty} \int_{[y,z]} f^{(N)}$.

Now notice that for each N we have $\int_{[y,z]} f^{(N)} = \sum_{n=0}^{N} \int_{[y,z]} c_n x^n$, so that $\int_{[y,z]} f^{(N)} = \sum_{n=0}^{\infty} \int_{[y,z]} c_n x^n$.

Denny: Cool! Then we can also differentiate $f(x) = \sum_{n=0}^{\infty} c_n x^n$ term by term! I mean $f(x) = \sum_{n=0}^{\infty} (c_n x^n)' = \sum_{n=1}^{\infty} n c_n x^{n-1}$.

Frank: Why?

Denny: By Theorem 3.7.1 that we covered in Module 53.

Differentiating power series term by term

Frank: Not so fast. You would need to prove first that if $\sum_{n=0}^{\infty} c_n x^n$ converges on $(-R, R)$, then $\sum_{n=1}^{\infty} n c_n x^{n-1}$ also converges on $(-R, R)$.

Theo: In fact, we can differentiate and integrate power series inside the interval $(-R, R)$. This is what Theorem 4.1.6(d)(e) of the textbook allow us to do. Denny's observation is part (d).

For integration, Bob has essentially given us a proof. And Frank is right, we need to examine the radius of convergence of the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$ to prove that differentiation term by term works.

Denny: Let's leave this for our classmates in the next module. Can we look at a specific example?

Cindy: How about the simplest one?

$f(x) := \sum_{n=0}^{\infty} x^n$, with $c_n = 1$ for all n.

Question C24.5: What is the radius of convergence R in Cindy's Example?

Differentiating Cindy's example term by term

Bob: $R = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}} = \frac{1}{\limsup_{n\to\infty} 1^{1/n}} = 1.$

This makes sense. Last semester we saw that $\sum_{n=0}^{\infty} x^n$ is divergent for $|x| \geq 1$. We also saw that for $x \in (-1,1)$ we have $f(x) := \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Denny: So when we differentiate term by term, as I have suggested, we obtain from the chain rule:
$f'(x) = (\frac{1}{1-x})' = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}.$

Frank: That's not as in the formula for power series!
The summation index should start with 0.

Cindy: Can we make a substitution $k = n-1$ and write Denny's formula as $\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k$?

Question C24.6: Would this work as Cindy suggested?

Alice: Excellent idea, Cindy!

Theo: And since the summation index is a dummy variable, we can also write $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$.
It does not matter which letter we use.

Cindy: Now let me integrate term by term. For $-1 < y < z < 1$ I will then get from the FTC: $\int_{[y,\, z]} \frac{1}{1-x} = -\ln(1-x)|_y^z$

Frank: You forgot about the absolute value.
It should be $\ln|1-x|$ instead of $\ln(1-x)$.

Cindy: But for $x \in [y, z]$ as above we have $1 - x > 0$, so $|1-x| = 1-x$.

Frank: Alright.

Cindy: Then:

$$-\ln(1-x)|_y^z = -\ln(1-z) - (-\ln(1-y)) = \ln(1-y) - \ln(1-z).$$

Frank: This is getting too messy. Let's keep it simple and only look at the special case $-1 < y < 0$ with $z = 0$.

Integrating Cindy's example term by term, continued

Denny: Yeah. Keep it simple.

Cindy: Then I get $\ln(1 - z) = \ln 1 = 0$. So we will have
$\ln(1 - y) = \int_{[y,\,0]} \frac{1}{1-x} = \sum_{n=0}^{\infty} \int_{[y,\,0]} x^n$.
By the FTC:
$\sum_{n=0}^{\infty} \int_{[y,\,0]} x^n = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}]_y^0 = \sum_{n=0}^{\infty} \frac{-1}{n+1} y^{n+1}$
so that we get
$\ln(1 - y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} \dots$

Bob: If we substitute $x := -y$ in the formula, it becomes
$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$.
We showed this for $x \in [0, 1)$. I wonder whether this formula also holds for $x \in (-1, 0)$, where $\ln(1 + x)$ is also defined and the series converges.

Denny: Let's leave this for our classmates in Module 57 and call it quits for today.