Lecture 11: Unions and intersections of indexed families of sets

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Companion to Advanced Calculus

Indexed families of sets

A set whose elements are themselves sets is often called a *family.*

So far we have applied set-theoretic operations on pairs of sets, or, more generally, finite families of sets. In this course we will often need to work with infinite families of sets and set-theoretic operations on them. For naming the sets in an infinite family, we would quickly run out of letters of the English alphabet. So we will use subscripts and work with *indexed families* of sets.

Example L11.1: Consider the following family of subsets of the real line:

- For $n \in \mathbb{Z}$, we let $A_n = [n, \infty)$.
- We let $\mathcal{A} := \{A_n : \ n \in \mathbb{Z}\} = \{[n, \infty) : \ n \in \mathbb{Z}\}$.

That family $\mathcal{A}$ contains infinitely many sets, but we still would like to form the union and the intersection of all these sets:

$A_0 \cup A_1 \cup \cdots \cup A_n \cup \ldots$ and $A_0 \cap A_1 \cap \cdots \cap A_n \cap \ldots$

This is permissible in axiomatic set theory, but we need a better notation to avoid using the ellipsis ...

Unions and intersections of indexed families

Definition L11.1: Let J be an index set, and let $\mathcal{C} = \{C_j : \ j \in J\}$ be a family of sets that is indexed by J. Then we define:

$\bigcup_{j \in J} C_j = \{x : \ \exists j \in J \ \ x \in C_j\}$.

$\bigcap_{j \in J} C_j = \{x : \ \forall j \in J \ \ x \in C_j\}$.

Note that $\bigcup_{j \in J} C_j$ is defined in terms of the existential quantifier, while $\bigcap_{j \in J} C_j$ is defined in terms of the universal quantifier.

For example, let $J = \mathbb{Z}$, $A_n = [n, \infty)$, and $\mathcal{A} := \{A_n : \ n \in \mathbb{Z}\}$. Let $x \in \mathbb{R}$. Then there exists an integer n such that $n \leq x$, so that $x \in [n, \infty) = A_n$. Thus $\bigcup_{n \in \mathbb{Z}} A_n = \mathbb{R}$.

Question L11.1: What is $\bigcap_{n \in \mathbb{Z}} A_n$?

Let $x \in \mathbb{R}$. Then there exists an integer n such that $n > x$, so that $x \notin [n, \infty) = A_n$. Thus $\bigcap_{n \in \mathbb{Z}} A_n = \emptyset$.

Restrictions on the use of indices

One can use this notation to define intersections and unions of subfamilies of an indexed family.

Again, let $J = \mathbb{Z}$, $A_n = [n, \infty)$, and $\mathcal{A} := \{A_n : \ n \in \mathbb{Z}\}$.
Let $F := \{3, 4, 5\} \subsetneq J$. Then

$\bigcup_{j \in F} = \bigcup_{n=3}^{5} A_n = A_3 \cup A_4 \cup A_5 = \ [3, \infty)$.

$\bigcap_{j \in F} = \bigcap_{n=3}^{5} A_n = A_3 \cap A_4 \cap A_5 = \ [5, \infty)$.

This is an example of an indexed family with empty intersection such that each finite subfamily has nonempty intersection. We will discuss this observation in more detail in Module 11.

Multiple subscripts and multiple operations

One can also index families with multiple subscripts.

Example L11.2: Let $I = J = \mathbb{N}$, and consider the following family $\mathcal{C} := \{C_{i,j} : \ i, j \in \mathbb{N}\}$ of subsets of $\mathbb{N}^{\mathbb{N}}$:

- For $i, j \in \mathbb{N}$, we let $C_{i,j} := \{f \in \mathbb{N}^{\mathbb{N}} : \ f(i) = j\}$

Let us examine the set $D := \bigcap_{j\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} C_{i,j} = \bigcap_{j\in\mathbb{N}} \left(\bigcup_{i\in\mathbb{N}} C_{i,j}\right)$.

The brackets give us an indication how we can unravel the meaning of this set: By working from inside out, that is to say, from right to left. Fix $j \in \mathbb{N}$, and let $D_j := \bigcup_{i\in\mathbb{N}} C_{i,j}$. Then $D_j = \{f \in \mathbb{N}^{\mathbb{N}} : \ \exists i \in \mathbb{N} \ \ f \in C_{i,j}\} = \{f \in \mathbb{N}^{\mathbb{N}} : \ \exists i \in \mathbb{N} \ \ f(i) = j\}$ is the set of all functions $f : \mathbb{N} \to \mathbb{N}$ that take on the value j for some argument. It follows that $D = \{f : \ \forall j \in \mathbb{N} \ \ f \in D_j\} = \{f \in \mathbb{N}^{\mathbb{N}} : \ \forall j \in \mathbb{N} \, \exists i \in \mathbb{N} \ \ f(i) = j\}$ is the set of all surjections $f : \mathbb{N} \to \mathbb{N}$.

- For $i, j \in \mathbb{N}$, we let $C_{i,j} := \{f \in \mathbb{N}^{\mathbb{N}} : \ f(i) = j\}$

Question L11.2: Find $E := \bigcup_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} C_{i,j} = \bigcup_{i\in\mathbb{N}} \left(\bigcap_{j\in\mathbb{N}} C_{i,j}\right)$.

For each $i \in \mathbb{N}$ we let $E_i := \bigcap_{j\in\mathbb{N}} C_{i,j}$. Then

$E_i = \{f \in \mathbb{N}^{\mathbb{N}} : \ \forall j \in \mathbb{N} \ \ f \in C_{i,j}\} = \{f \in \mathbb{N}^{\mathbb{N}} : \ \forall j \in \mathbb{N} \ \ f(i) = j\}$.

Then for all $i \in \mathbb{N}$ the set $E_i = \emptyset$, since no function can take more then one value for input i.

Thus also $E = \{f : \ \exists i \in \mathbb{N} \ \ f \in E_i\} = \{f : \ \exists i \in \mathbb{N} \, \exists i \ f \in \emptyset\} = \emptyset$.

Notice how working from inside out (that is, from right to left) and translating definitions of sets into logical expressions with quantifiers helped us unravel the definitions of the sets D and E in this example.

Take-home message

Let J be an index set, and let $\mathcal{C} = \{C_j : \ j \in J\}$ be a family of sets that is indexed by J. Then we define:

$\bigcup_{j \in J} C_j = \{x : \ \exists j \in J \ \ x \in C_j\}$.

$\bigcap_{j \in J} C_j = \{x : \ \forall j \in J \ \ x \in C_j\}$.

Note that $\bigcup_{j \in J} C_j$ is defined in terms of the existential quantifier, while $\bigcap_{j \in J} C_j$ is defined in terms of the universal quantifier.

When we have multiple subscripts, we can also form sets like

$\bigcup_{i \in I} \bigcap_{j \in J} C_{i,j}$ or $\bigcap_{j \in J} \bigcup_{i \in I} C_{i,j}$.

Here the procedure is to first fix the subscript that varies on the left and form the union or intersection over the other subscript.

When unions and intersections are combined, the order of operations matters.