Lecture 15: Absolute values, and ε-closeness, and integer powers of real numbers

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Companion to Advanced Calculus

Review: The state of the course

Up to now, this course has focused on important tools for studying advanced calculus: Formal logic, development of rigorous mathematical theories from axioms, techniques for proving theorems, and tools from set theory.

We also need certain types of mathematical objects: Functions, relations, the number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ together with arithmetic operations on them and their ordering.

We have already studied functions and important properties of them. Although the textbook does not do this, this development can actually be formalized in axiomatic set theory by treating functions as special kinds of relations.

We also have seen how axiomatic development of the number systems $(\mathbb{N}, +, \cdot, 0, <)$, $(\mathbb{Z}, +, \cdot, 0, 1, <)$, $(\mathbb{Q}, +, \cdot, 0, 1, <)$ and their properties can be carried out axiomatically.

After we do the same for $(\mathbb{R}, +, \cdot, 0, 1, <)$, we will be ready to actually study Advanced Calculus.

Constructing the reals: The textbook's exposition

It remains to construct the number system $(\mathbb{R}, +, \cdot, 0, 1, <)$ and derive its most important properties axiomatically.

This is what the textbook does in Chapter 5.

The textbook's exposition of this material has several drawbacks:

First of all, it further delays the moment when we can actually start studying advanced calculus.

Second, the textbook construction of the reals requires the notion of a *Cauchy sequence* that is a fairly advanced concept of advanced calculus itself. This concept is easier to understand if we first study the related notion of a *convergent sequence.*

Third, in order to avoid circularity, the textbook needs to define a number of important notions twice: First only for the rationals, and then later for all reals. This further slows down the exposition.

Constructing the reals: Our exposition

We will take a different approach that is more intuitive and straightforward: We will temporarily assume that we have $(\mathbb{R}, +, \cdot, 0, 1, <)$ already in place, with all the properties that are familiar to us from calculus. In particular, we treat $\mathbb{R}$ as a number line, linearly ordered by $<$, and $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ as usual.

Starting with this lecture, we develop some notions of advanced calculus based on these intuitions. This will work fine initially, but already in the next lecture we will run into a major difficulty with this approach.

After temporarily suspending our concern about this difficulty and developing additional notions of advanced calculus, we will then see how one can rigorously define the set $\mathbb{R}$ of real numbers so that the issue gets resolved. After that, we will be able to fully focus on the development of advanced calculus.

The absolute value of a real number

The textbook gives the following definitions only for rational numbers, but they work for all reals:

Definition 4.3.1: (Absolute value) If x is a real number, the *absolute value* $|x|$ *of* x is defined as follows.

- If x is positive, then $|x| := x$.
- If x is negative, then $|x| := -x$.
- If x is zero, then $|x| := 0$.

Definition 4.3.2: (Distance) Let x and y be real numbers. The quantity $|x - y|$ is called *the distance between* x *and* y and is sometimes denoted $d(x, y)$. Thus $d(x, y) := |x - y|$.

Proposition 4.3.3 of the textbook then lists some properties of the absolute value and the distance function that will already be familiar to you from calculus. We will explore some of them in Module 15.

Basic properties of the distance function

The distance function is a function on $\mathbb{R} \times \mathbb{R}$ that takes real values and has the following properties:

(i) (Non-degeneracy of distance) We have $d(x, y) \geq 0$.
Also, $d(x, y) = 0$ if and only if $x = y$.

(ii) (Symmetry of distance) $d(x, y) = d(y, x)$.

(iii) (Triangle inequality for distance) $d(x, z) \leq d(x, y) + d(y, z)$.

If X is any set, then any function $d : X \times X \to \mathbb{R}$ that satisfies properties (i), (ii), and (iii) above is called a *metric on X.*

In MATH4/5302 we will explore metrics on other sets X.

For now let us just observe that a small distance between x and y means that x and y are "close" in some sense. Your textbook formalizes this notion in a way that we will present on the next few slides. Although this terminology is not standard in the literature, it will be very helpful in guiding our thinking about a number of concepts and proofs in advanced calculus.

The definition of ε-closeness of real numbers

Definition 4.3.4: (ε-closeness). Let $\varepsilon > 0$ be a real number, and let x, y be real numbers. We say that y is *ε-close to x* iff we have $d(y, x) \leq \varepsilon$, that is, when $|x - y| \leq \varepsilon$.

- The textbook version defines this notion only for rational numbers, but it works just fine for reals.
- ε-closeness can be understood as a relation R_ε on $\mathbb{R}$, where $xR_\varepsilon y$ iff y is ε-close to x.
- For $\varepsilon \neq \varepsilon'$ we get different relations $R_\varepsilon \neq R_{\varepsilon'}$.

Question L15.1: Fix $\varepsilon > 0$. Is the relation R_ε reflexive?

Yes.

Question L15.2: Fix $\varepsilon > 0$. Is the relation R_ε symmetric?

Yes.

Question L15.3: Fix $\varepsilon > 0$. Is the relation R_ε transitive?

An important property of ε-closeness

No. For any given $\varepsilon > 0$, the relation R_ε is not transitive.

Consider, for example, $\varepsilon := 1.7$. The numbers 1 and 2 are 1.7-close, and so are 2 and 3, but 1 and 3 are not 1.7-close.

Thus ε-closeness is not an equivalence relation.

However, it has the following property that is weaker than transitivity:

- Let $\varepsilon, \delta > 0$. If x is ε-close to y, and y is δ-close to z, then x and z are $(\varepsilon + \delta)$-close.

To see why this property holds, consider $x, y, z \in \mathbb{R}$ such that $d(x, y) \leq \varepsilon$ and $d(y, z) \leq \delta$.

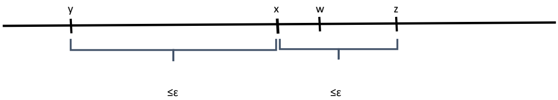
Then by the triangle inequality for the distance function,

$d(x, z) \leq d(x, y) + d(y, z) \leq \varepsilon + \delta$,

so that x and z are $(\varepsilon + \delta)$-close.

Another important property of ε-closeness

- Let $\varepsilon > 0$. If y and z are both ε-close to x, and w is between y and z (i.e., $y \leq w \leq z$ or $z \leq w \leq y$), then w is also ε-close to x.



Our picture illustrates this property for one special case of how x, y, z, w are related by $\leq$.

ε-closeness, absolute values, and intervals

We will explore additional properties of ε-closeness in Module 16.

Let us now explicitly state two observations that we will use frequently throughout the remainder of the course:

Proposition L15.1: Let x, y, ε be real numbers such that $\varepsilon > 0$. Then the following are equivalent:

- x and y are ε-close,
- $|x - y| \leq \varepsilon$,
- $x - \varepsilon \leq y \leq x + \varepsilon$,
- $y \in [x - \varepsilon, x + \varepsilon]$.

Proposition L15.2: Let x, y be real numbers. Then x and y are ε-close for every $\varepsilon > 0$ if, and only if, $x = y$.

In symbols: $x = y \;\text{ iff }\; \forall \varepsilon > 0 \;\; |x - y| \leq \varepsilon$.

Raising a real number to a natural exponent

Definition 4.3.9: (Exponentiation to a natural number) Let x be a real number. To raise x to the power 0, we define $x^0 := 1$; in particular we define $0^0 := 1$.

Now suppose inductively that x^n has been defined for some natural number n, then we define $x^{n+1} := x^n x$.

This definition is very familiar to us, except that the textbook here defines $0^0 := 1$ instead of treating this power us an undetermined quantity as is more common in the literature. The primary motivation here is smoother axiomatic development. We can use the textbook definition for most purposes and revert to treating 0^0 as undetermined by explicitly saying so in situations where this actually matters.

Note that Definition 4.3.9 is an example of a *recursive definition.*

Raising a real number to an integer exponent

Definition 4.3.11: (Exponentiation to a negative number). Let x be a non-zero real number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.

Notice that the textbook gives Definitions 4.3.9 and 4.3.11 only for the case when x is rational. But they work in just the same way for all real numbers x.

Note that x^n is now defined for all $x \neq 0$ and all integers n. However, when $x = 0$, only the powers x^n for natural numbers $n \geq 0$ are defined, as we cannot divide by x^n if $x^n = 0^n = 0$.

Properties of integer powers of reals

Proposition 4.3.12: (Properties of exponentiation) Let x, y be nonzero real numbers, and let n, m be integers.

(a) We have $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.

(b) If $x \geq y > 0$, then
- $x^n \geq y^n > 0$ if n is positive, and
- $0 < x^n \leq y^n$ if n is negative.

(c) If $x, y > 0, n \neq 0$, and $x^n = y^n$, then $x = y$.

(d) We have $|x^n| = |x|^n$.

These properties are already familiar to us from precalculus.

Note that in part (c) the assumptions $x, y > 0, n \neq 0$ are needed. For example, $x := -3 \neq y := 3$, but $(-3)^2 = 3^2 = 9$. Similarly, $x := 4 \neq y := 5$ are both positive, but $4^0 = 5^0 = 1$.

However, when n is an odd integer, then the implication $x^n = y^n \Longrightarrow x = y$ holds whenever $x \neq 0 \neq y$.