Lecture 22: The extended real number system; suprema and infima of sequences

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Companion to Advanced Calculus

Adding two symbols to the real numbers

In Section 6.2, the textbook defines *extended real number system* $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$.

Here $-\infty$ and $+\infty$ are just symbols; they are *not* real numbers. We will often simply write ∞ instead of $+\infty$.

For some, but not all, uses of these symbols, we can think of $+\infty$ as representing an unspecified very large positive number and of $-\infty$ as representing an unspecified very large negative number.

We can then extend the linear order relation $\leq$ and the corresponding strict linear order relation $<$ on $\mathbb{R}$ to a linear order relation and a strict linear order relation on $\mathbb{R}^*$ in the obvious way, so that $-\infty < x < +\infty$ for all $x \in \mathbb{R}$.

Question L22.1: Can we also extend addition to $\mathbb{R}^*$?

Some arithmetic on $\mathbb{R}^*$, sort of

No.

If we think about the new symbols as suggested above, the quantity $-\infty + \infty$ could be anything; large, small, or even 0.

Thus we need to leave $-\infty + \infty$ undefined.

But we can meaningfully define the following:

- $-(+\infty) := -\infty$ and $-(-\infty) := +\infty$.
- $x + (+\infty) = (+\infty) + x := +\infty$ for all $x \in \mathbb{R} \cup \{+\infty\}$.
- $x + (-\infty) := (-\infty) + x := -\infty$ for all $x \in \mathbb{R} \cup \{-\infty\}$.

Review: Least upper bounds

Definition 5.5.1 (Upper bound). Let E be a subset of $\mathbb{R}^*$, and let M be a real number. We say that M is an *upper bound* for E, iff we have $x \leq M$ for every element $x \in E$.

Definition 5.5.5: (Least upper bound). Let E be a subset of $\mathbb{R}^*$, and M be a real number. We say that M is a *least upper bound* for E iff

(a) M is an upper bound for E, and also

(b) any other upper bound M' for E must be larger than or equal to M.

Note that we used here versions of these definitions that allow E to be a set of extended real numbers. But the bound M must always be a real number.

One can define *lower bounds* and *greatest lower bounds* analogously; we will do this in Module 22.

The supremum and the infimum of a set

If they exist for a given set $E \subseteq \mathbb{R}$, the least upper bound and the greatest lower bound must be unique. But they do not exist for every set. The concepts of a *supremum* $\sup(E)$ and *infimum* $\inf(E)$ of a set $E \subseteq \mathbb{R}^*$ allow us to conveniently talk about a least upper bound or greatest lower bound without having to say all the time "if it exists."

Definition L22.1: Let $E \subseteq \mathbb{R}^*$.

- If $\emptyset \neq E$ and E has some upper bound, we define $\sup(E)$ to be the least upper bound of E.
- If $\emptyset \neq E$ and E has no upper bound, or if $+\infty \in E$, we set $\sup(E) := +\infty$.
- If $\emptyset \neq E$ and E has some lower bound, we define $\inf(E)$ to be the greatest lower bound of E.
- If $\emptyset \neq E$ and E has no lower bound, or if $-\infty \in E$, we set $\inf(E) := -\infty$.
- If $E = \emptyset$, we set $\sup(E) := -\infty$ and $\inf(E) := +\infty$.

Example L22.1: Let $E := \{\frac{1}{n} : \ n \in \mathbb{N}\}$.

Then $\sup(E) = 1$, which is also the largest element of E.

Moreover, $\inf(E) = 0$.
This is the greatest lower bound of E, but not an element of E, as E does not have a smallest element.

Example L22.2: Let $E := \mathbb{N}$.

Question L22.2: Find $\inf(E)$ and $\sup(E)$.

$\inf(E) = 0$ and $\sup(E) = +\infty$.

Infima and suprema of sequences

Definition 6.3.1: (Sup and inf of sequences) Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Then we define $\sup(a_n)_{n=m}^{\infty}$ to be the supremum of the set $\{a_n : \ n \geq m\}$, and $\inf(a_n)_{n=m}^{\infty}$ to the infimum of the same set $\{a_n : \ n \geq m\}$.

Note that this is not a new concept, only a new notation and new names for $\sup\{a_n : \ n \geq m\}$ and $\inf\{a_n : \ n \geq m\}$ that are already previously defined quantities. It is important to keep in mind though that the *sequence* $(a_n)_{n=m}^{\infty}$ is a different kind of mathematical object (a function) than the *set* $\{a_n : \ n \geq m\}$ of values of this function (its range).

Following the textbook, we will sometimes write the quantities $\sup(a_n)_{n=m}^{\infty}$ and $\inf(a_n)_{n=m}^{\infty}$ as $\sup_{n \geq m} a_n$ and $\inf_{n \geq m} a_n$, respectively.

Infima and suprema of sequences: Examples

Example L22.3: Let $(a_n)_{n=1}^{\infty} := \left(\frac{1}{n}\right)_{n=1}^{\infty}$.

Then $\{a_n : \ n \geq 1\}$ is the set E of Example L22.1.

It follows that $\inf_{n \geq 1} a_n = 0$ and $\sup_{n \geq 1} a_n = 1$.

Recall from Module 17 that a sequence $(a_n)_{n=m}^{\infty}$

increases without bound iff $\forall M \in \mathbb{R}\, \exists N\, \forall n \geq N \quad a_n > M$ and

decreases without bound iff $\forall M \in \mathbb{R}\, \exists N\, \forall n \geq N \quad a_n < M$.

We will from now on write $\lim_{n\to\infty} a_n = \infty$ when a sequence increases without bound and $\lim_{n\to\infty} a_n = -\infty$ when a sequence decreases without bound. However, it needs to be kept in mind that such sequences are *divergent*, as $-\infty$ and ∞ are not a real numbers and do not count as limits.

Question L22.3: Does $\lim_{n\to\infty} a_n = \infty$ mean the same thing as $\sup_{n \geq 1} a_n = \infty$?

No. $\lim_{n\to\infty} a_n = \infty$ implies, but is not equivalent to, $\sup_{n \geq 1} a_n = \infty$.

$(a_n)_{n=1}^{\infty} := ((-1)^n n)_{n=1}^{\infty}$ is an example of a sequence that does not increase without bound, while $\sup_{n \geq 1} a_n = \infty$.