Lecture 23: Monotone sequences, limsup and liminf of sequences

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Companion to Advanced Calculus
A sequence \((a_n)_{n=m}^{\infty}\) of elements of \(\mathbb{R}^*\) is:

- **increasing** if \(a_n \leq a_{n+1}\) for all \(n \geq m\),
- **strictly increasing** if \(a_n < a_{n+1}\) for all \(n \geq m\),
- **decreasing** if \(a_n \geq a_{n+1}\) for all \(n \geq m\),
- **strictly decreasing** if \(a_n > a_{n+1}\) for all \(n \geq m\),
- **monotone** if it is increasing or decreasing,
- **strictly monotone** if it is either strictly increasing or strictly decreasing.

**Example L23.1:** Any constant sequence is simultaneously increasing and decreasing, but neither strictly increasing nor strictly decreasing.

**Example L23.2:** The sequence \((a_n)_{n=1}^{\infty} := ((-1)^n)_{n=1}^{\infty}\) is not monotone. However, notice that this sequence is bounded.
Example L23.3: Let $a_n := \frac{1}{n}$ for all $n \geq 1$. Then $(a_n)_{n=1}^{\infty}$ is strictly decreasing, convergent, and $\lim_{n \to \infty} a_n = 0 = \inf(a_n)_{n=1}^{\infty}$.

For every $N \geq 1$, let $a_N^+ := \sup(a_n)_{n=N}^{\infty}$ and $a_N^- := \inf(a_n)_{n=N}^{\infty}$.

Then in this example: $a_N^+ = a_N = \frac{1}{N}$ and $a_N^- = 0$ for all $N \geq 1$. 
**Example L23.4:** Let \( b_n := \arctan n \) for all \( n \in \mathbb{N} \).
Then \( (b_n)_{n=0}^\infty \) is strictly increasing, convergent, and
\[
\lim_{n \to \infty} b_n = \frac{\pi}{2} = \sup(b_n)_{n=1}^\infty \approx 1.5708.
\]

For every \( N \in \mathbb{N} \), let \( b^+_N := \sup(b_n)_{n=N}^\infty \) and \( b^-_N := \inf(b_n)_{n=N}^\infty \).

**Question L23.1:** What are \( b^+_N \) and \( b^-_N \) in this example?

\( b^+_N = b_N = \frac{\pi}{2} \) and \( b^-_N = b_N = \arctan N \) for all \( N \in \mathbb{N} \).
Examples 3 and 4 generalize

Not all monotone sequences converge. For example, the sequence \((n)_{n=0}^\infty\) is strictly increasing, but divergent. And as Example L23.2 shows, non-monotone bounded sequences may also diverge.

But *bounded monotone* sequences always converge:

**Proposition 6.3.8:** (Monotone bounded sequences converge) Let \((a_n)_{n=m}^\infty\) be a bounded sequence of real numbers and let \(M > 0\) be a bound in \(\mathbb{R}\) for this sequence.

(a) If \((a_n)_{n=m}^\infty\) is increasing, then it is convergent, and in fact
\[
\lim_{n \to \infty} a_n = \sup(a_n)_{n=m}^\infty \leq M.
\]

(b) If \((a_n)_{n=m}^\infty\) is decreasing, then it is convergent, and in fact
\[
\lim_{n \to \infty} a_n = \inf(a_n)_{n=m}^\infty \geq -M.
\]

The above version of this result gives more information than the textbook version, which is restricted to our part (a).

We will formally prove this proposition in Module 23.
Example L23.5: Let $c_n := \sin n$ for all $n \in \mathbb{N}$. Then $c_0 = 0$, $c_1 \approx 0.8415$, $c_2 \approx 0.9093$, $c_3 \approx 0.1411$, $c_4 \approx -0.7568$. This sequence is not monotone. It is bounded, since $|c_n| \leq 1$ for every $n \in \mathbb{N}$, but it is not convergent.

For every $N \in \mathbb{N}$, let $c_N^+ := \sup(c_n)^\infty_{n=N}$ and $c_N^- := \inf(c_n)^\infty_{n=N}$.

Question L23.2: What appear $c_N^+$ and $c_N^-$ to be in this example? Here $c_N^+ = 1$ and $c_N^- = -1$ for all $N \in \mathbb{N}$. 
The sequences \((a_N^+)_{N=m}^\infty\) and \((a_N^-)_{N=m}^\infty\)

Let \((a_n)_{n=m}^\infty\) be any sequence or reals. For every \(N \geq m\) define
\[ a_N^+ := \sup(a_n)_{n=N}^\infty \quad \text{and} \quad a_N^- := \inf(a_n)_{n=N}^\infty. \]
In general, \(a_N^+\) and \(a_N^-\) are in \(\mathbb{R}^*\).
But when \((a_n)_{n=m}^\infty\) is bounded, they are always real numbers.

**Question L23.3:** What can we say about the monotonicity properties of the sequence \((a_N^+)_{N=m}^\infty\)?

This sequence will be decreasing, but not always strictly decreasing.

For the sequence \((a_n)_{n=1}^\infty = (1/n)_{n=1}^\infty\) of Example L23.3 we have
\[ a_N^+ = a_N = \frac{1}{n} \quad \text{for all} \quad N \geq 1 \quad \text{so that} \quad (a_N^+)_{N=1}^\infty \quad \text{is strictly decreasing.} \]

For the sequence \((b_n)_{n=0}^\infty\) of Example L23.4 we have \(b_N^+ = \frac{\pi}{2}\) for all \(N \geq 0\). Thus \((b_N^+)_{N=1}^\infty\) is decreasing, but not strictly decreasing.

Similarly, the sequence \((a_N^-)_{N=m}^\infty\) will always be increasing,
but it may or may not be strictly increasing.
Definition 6.4.6: (Limit superior) Suppose that $(a_n)_{n=m}^\infty$ is a sequence. We define a new sequence $(a_N^+)_{N=m}^\infty$ by the formula

$$a_N^+ := \sup(a_n)_{n=N}^\infty.$$ 

More informally, $a_N^+$ is the supremum of all the elements in the sequence from $a_N$ onwards.

We then define the **limit superior** of the sequence $(a_n)_{n=m}^\infty$, denoted $\limsup_{n\to\infty} a_n$, by the formula

$$\limsup_{n\to\infty} a_n := \inf(a_N^+)_{N=m}^\infty.$$ 

Notice that if $(a_n)_{n=m}^\infty$ is bounded, then

- $\sup(a_n)_{n=N}^\infty \in \mathbb{R}$,
- $(a_N^+)_{N=m}^\infty$ is bounded and decreasing,
- $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} a_N^+$.

The last item follows from part (b) of Proposition 6.3.8.
Definition 6.4.6: (Limit inferior) Suppose that \((a_n)_{n=m}^{\infty}\) is a sequence. We define a new sequence \((a_{-N})_{N=m}^{\infty}\) by the formula
\[
a_{-N} := \inf(a_n)_{n=N}^{\infty}.
\]
More informally, \(a_{-N}\) is the infimum of all the elements in the sequence from \(a_N\) onwards.

We then define the limit inferior of the sequence \((a_n)_{n=m}^{\infty}\), denoted \(\liminf_{n \to \infty} a_n\), by the formula
\[
\liminf_{n \to \infty} a_n := \sup(a_{-N})_{N=m}^{\infty}.
\]

Notice that if \((a_n)_{n=m}^{\infty}\) is bounded, then

- \(\inf(a_n)_{n=N}^{\infty} \in \mathbb{R}\),
- \((a_{-N})_{N=m}^{\infty}\) is bounded and increasing,
- \(\liminf_{n \to \infty} a_n = \lim_{N \to \infty} a_{-N}\).

The last item follows from part (a) of Proposition 6.3.8.
Example L23.6: Let \((a_n)_{n=1}^\infty = (-n)_{n=1}^\infty\).
Then \(a^-_N = -\infty\) and \(a^+_N = -N\) for all \(N \geq 1\).
Thus \(\liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = -\infty\).

For the sequence \((a_n)_{n=1}^\infty := ((-1)^n)_{n=1}^\infty\) of Example L23.2
we have \(a^-_N = -1\) and \(a^+_N = 1\) for all \(N \geq 1\).
Thus \(\liminf_{n \to \infty} a_n = -1\) and \(\limsup_{n \to \infty} a_n = 1\).

Similarly, for the sequence \((c_n)_{n=0}^\infty := (\sin n)_{n=0}^\infty\) of Example L23.5
we have \(\liminf_{n \to \infty} c_n = -1\) and \(\limsup_{n \to \infty} c_n = 1\).

Question L23.4: Find \(\liminf_{n \to \infty} a_n\) and \(\limsup_{n \to \infty} a_n\)
for the sequence \((a_n)_{n=1}^\infty := (\frac{1}{n})_{n=1}^\infty\) of Example L23.3.
\(\liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = \lim_{n \to \infty} a_n = 0\).

Also, for the sequence \((b_n)_{n=0}^\infty := (\arctan n)_{n=0}^\infty\) of Example L23.4
we get \(\liminf_{n \to \infty} b_n = \limsup_{n \to \infty} b_n = \lim_{n \to \infty} b_n = \frac{\pi}{2}\).
We defined several types of monotone sequences: increasing, decreasing, strictly increasing, and strictly increasing ones.

Bounded monotone sequences are always convergent.

With every sequence \((a_n)_{n=m}^{\infty}\) of real numbers we associate two sequences \((a_N^+)_{N=m}^{\infty}\) and \((a_N^-)_{N=m}^{\infty}\) of extended real numbers defined by
\[
a_N^+ := \sup(a_n)_{n=N}^{\infty} \quad \text{and} \quad a_N^- := \inf(a_n)_{n=N}^{\infty}.
\]
The sequence \((a_N^-)_{N=m}^{\infty}\) is always increasing and the sequence \((a_N^+)_{N=m}^{\infty}\) is always decreasing.

Two extended real numbers, called the limit inferior and limit superior, respectively, of \((a_n)_{n=m}^{\infty}\) were defined as follows:
\[
\liminf_{n \to \infty} a_n := \sup(a_N^-)_{N=m}^{\infty},
\]
\[
\limsup_{n \to \infty} a_n := \inf(a_N^+)_{N=m}^{\infty}.
\]
Take-home message, continued

For \textit{unbounded} sequences \((a_n)_{n=m}^{\infty}\), it is possible that 
\[
\liminf_{n \to \infty} a_n = -\infty \quad \text{or} \quad \limsup_{n \to \infty} a_n = -\infty
\]
or 
\[
\liminf_{n \to \infty} a_n = \infty \quad \text{or} \quad \limsup_{n \to \infty} a_n = \infty.
\]

When the sequence \((a_n)_{n=m}^{\infty}\) is \textit{bounded}, then:

- \(a^-_N\) and \(a^+_N\) are real numbers for all \(N \geq m\),
- \(\liminf_{n \to \infty} a_n\) is a real number,
- \(\limsup_{n \to \infty} a_n\) is a real number,
- \(\liminf_{n \to \infty} a_n = \lim_{N \to \infty} a^-_N\),
- \(\limsup_{n \to \infty} a_n = \lim_{N \to \infty} a^+_N\).