Lecture 29: Subsets of the real line

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Companion to Advanced Calculus

Intervals of reals

Definition 9.1.1: (Intervals) Let $a, b \in \mathbb{R}^*$ be extended real numbers. We define the *closed interval* $[a, b]$ by

$[a, b] := \{x \in \mathbb{R}^* : \; a \leq x \leq b\}$,

the *half-open intervals* $[a, b)$ and $(a, b]$ by

$[a, b) := \{x \in \mathbb{R}^* : \; a \leq x < b\}; \quad (a, b] := \{x \in \mathbb{R}^* : \; a < x \leq b\}$,

and the *open intervals* (a, b) by

$(a, b) := \{x \in R^* : \; a < x < b\}$.

We call a the *left endpoint* of these intervals, and b the *right endpoint.*

- $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^* = [-\infty, +\infty]$ are intervals.
- Intervals with both endpoints in $\mathbb{R}$ are *bounded intervals.*
- $[a, a] = \{a\}$ for all $a \in \mathbb{R}^*$. These are *degenerate intervals.*
- When $b < a$, then $[a, b] = [a, b) = (b, a] = (a, b) = \emptyset$. These are also *degenerate intervals.*

Definition 9.1.22: (Bounded sets) A subset X of the real line is said to be *bounded* if we have $X \subseteq [-M, M]$ for some real number $M > 0$.

A set of reals that is not bounded will be called *unbounded.*

- All bounded intervals are bounded sets.
- $X \subseteq \mathbb{R}$ is bounded iff $\exists M > 0\, \forall x \in X \quad |x| \leq M$.
- The interval $[0, +\infty)$ is unbounded.
- The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ are all unbounded.

Question L29.1: Suppose that (a, b) is an interval such that at least one of the endpoints a, b is not in $\mathbb{R}$. Is then (a, b) always unbounded?

This is only the case when $a \leq b$. The degenerate interval $(\infty, -\infty)$ is empty and hence bounded.

Why do we need to add the endpoints to make an open interval (a, b) "closed"? Adherent points

Definition 9.1.5: (ε-adherent points) Let X be a subset of $\mathbb{R}$, let $\varepsilon > 0$, and let $x \in \mathbb{R}$. We say that x is *ε-adherent to X* iff there exists a $y \in X$ which is ε-close to x. In symbols:

$\exists y \in X \quad |x - y| \leq \varepsilon.$

Question L29.2: What is the set of 0.5-adherent points of $(1, 2]$?

This is the interval $(0.5, 2.5]$.

Definition 9.1.8: (Adherent points) Let X be a subset of $\mathbb{R}$, and let $x \in \mathbb{R}$. We say that x is an *adherent point* of X iff it is ε-adherent to X for every $\varepsilon > 0$. In symbols:

$\forall \varepsilon > 0 \, \exists y \in X \quad |x - y| \leq \varepsilon.$

Question L29.3: What is the set of adherent points of $(1, 2]$?

This is the closed interval $\bigcap_{\varepsilon > 0} (1 - \varepsilon, 2 + \varepsilon] = [1, 2]$.

Adherent points: Other definitions and more examples

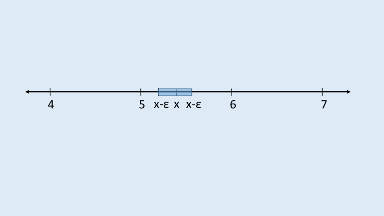
Observation L29.1: Let $X \subseteq \mathbb{R}$, and let $x \in \mathbb{R}$. Then the following conditions are all equivalent:

(a) x is an adherent point of X.
(b) $\forall \varepsilon > 0 \, \exists y \in X \;\; |x - y| \leq \varepsilon$.
(c) $\forall \varepsilon > 0 \;\; [x - \varepsilon, x + \varepsilon] \cap X \neq \emptyset$.
(d) $\forall \varepsilon > 0 \, \exists y \in X \;\; |x - y| < \varepsilon$.
(e) $\forall \varepsilon > 0 \;\; (x - \varepsilon, x + \varepsilon) \cap X \neq \emptyset$.

Let us look at some examples:

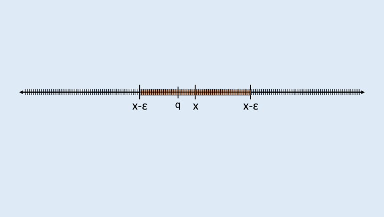
- Every $x \in X$ is an adherent point of X:
  Here we can take $y = x$ in point (b) above.
- The set of adherent points of $\emptyset$ is $\emptyset$.
- The set of adherent points of $\mathbb{Z}$ is $\mathbb{Z}$ itself:
  For each $x \notin \mathbb{Z}$, if we let $\varepsilon := \frac{1}{2} \min\{x - \lfloor x \rfloor, \lfloor x \rfloor + 1 - x\}$, then $[x - \varepsilon, x + \varepsilon] \cap \mathbb{Z} = \emptyset$.

Adherent points: More examples



Question L29.4: What is the set of adherent points of $\mathbb{Q}$?

The entire set $\mathbb{R}$. This follows from the fact that for every $x \in \mathbb{R}$ and $\varepsilon > 0$ there exists $q \in \mathbb{Q}$ such that $x - \varepsilon < q < x + \varepsilon$.



The closure of a set of reals

Definition 9.1.10: (Closure) Let X be a subset of $\mathbb{R}$. The *closure* of X, sometimes denoted $\overline{X}$, is defined to be the set of all the adherent points of X.

- Let $a, b \in \mathbb{R}$ with $a < b$. Then $\overline{(a,b)} = \overline{[a,b)} = \overline{(a,b]} = \overline{[a,b]} = [a,b]$.
- $\overline{\emptyset} = \emptyset$.
- $\overline{\mathbb{N}} = \mathbb{N}$, $\overline{\mathbb{Z}} = \mathbb{Z}$, $\overline{\mathbb{R}} = \mathbb{R}$, while $\overline{\mathbb{Q}} = \mathbb{R}$.
- We always have $X \subseteq \overline{X}$.

Definition 9.1.15: A subset $E \subset \mathbb{R}$ is said to be *closed* if $E = \overline{E}$, or, in other words, when E contains all of its adherent points.

Thus $\emptyset$, $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and all closed intervals $[a, b]$ are closed, while $\mathbb{Q}$ and nondegenerate open and half-open bounded intervals $(a, b), [a, b), (a, b]$ are not closed.

A characterization of adherent points

Lemma 9.1.14: Let $X \subseteq \mathbb{R}$, and let $x \in \mathbb{R}$. Then x is an adherent point of X if, and only if, there exists a sequence $(a_n)_{n=0}^{\infty}$, consisting entirely of elements of X, which converges to x.

Proof: Suppose $(a_n)_{n=0}^{\infty}$ is a sequence of elements of X, and $\lim_{n\to\infty} a_n = x$.

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\forall n \geq N \;\; |a_n - x| \leq \varepsilon$.

By letting $y := a_N \in X$, we see that x is ε-adherent to X.

As ε can be any positive real, it follows that x is adherent to X.

Now assume that x is an adherent point for X.

Then for each $n \in \mathbb{N}$ we can pick $a_n \in X$ such that $|x - a_n| \leq \frac{1}{n+1}$.

Let $\varepsilon > 0$. By the Archimedian property, there exists $N > 0$ such that $\frac{1}{N+1} < \varepsilon$.

Then $\forall n \geq N \;\; |a_n - x| \leq \varepsilon$, which shows that $\lim_{n\to\infty} a_n = x$. $\square$

Corollary 9.1.17: Let X be a subset of $\mathbb{R}$. If X is closed, and $(a_n)_{n=0}^{\infty}$ is a convergent sequence consisting of elements of X, then $\lim_{n\to\infty} a_n$ also lies in X.
Conversely, if it is true that every convergent sequence $(a_n)_{n=0}^{\infty}$ of elements of X has its limit in X as well, then X is necessarily closed.

This result explains our terminology: A set of real numbers is closed if, and only if, it is closed under limits of convergent sequences whose elements are all in X.

It also gives another explanation why $\mathbb{R}$ must the closure of $\mathbb{Q}$:
In Chapter 5 we *defined* the reals as the limits of Cauchy (and hence convergent) sequences of rational numbers.
Thus Corollary 9.1.17 implies that $\mathbb{R}$ is the set of all adherent points of $\mathbb{Q}$, that is, $\mathbb{R} = \overline{\mathbb{Q}}$.

Limit points of a set $X \subseteq \mathbb{R}$

Adherent points of a set $X \subseteq \mathbb{R}$ come in two flavors:

Definition 9.1.18: (Limit points) Let X be a subset of the real line. We say that x is a *limit point* (or a *cluster point*) of X iff it is an adherent point of $X \backslash \{x\}$. In symbols:

$\forall \varepsilon > 0 \, \exists y \in X \quad (|x - y| \leq \varepsilon \wedge y \neq x)$.

We say that x is an *isolated point* of X if $x \in X$ and

$\exists \varepsilon > 0 \, \forall y \in X \quad (|x - y| > \varepsilon \vee y = x)$.

- Let $X := (0, 2] \cup \{3\}$. Then $x := 3$ is an isolated point of X, while $[0, 2]$ is the set of limit points of X.
- For every $X \subseteq \mathbb{R}$, each $x \in \overline{X} \backslash X$ must be a limit point of X.
- The closure $\overline{X}$ of any $X \subseteq \mathbb{R}$ is obtained by adding to X the set of all its limit points.

Characterizations of isolated points

Observation L29.2: Let $X \subseteq \mathbb{R}$ and let $x \in \mathbb{R}$.

Then the following conditions are all equivalent:

(a) x is an isolated point of X.

(b) $\exists \varepsilon > 0 \, \forall y \in X \backslash \{x\} \; |x - y| > \varepsilon$.

(c) $\forall \varepsilon > 0 \; X \cap [x - \varepsilon, x + \varepsilon] = \{x\}$.

(d) $\exists \varepsilon > 0 \, \forall y \in X \backslash \{x\} \; |x - y| \geq \varepsilon$.

(e) $\forall \varepsilon > 0 \; X \cap (x - \varepsilon, x + \varepsilon) = \{x\}$.

(f) Every sequence $(y_n)_{n=0}^{\infty}$ of elements of X with $\lim_{n \to \infty} y_n = x$ must be eventually constant, so that $\exists N \geq 0 \, \forall n \geq N \; y_n = x$.

If we think as ε representing 6 feet, then x will be an isolated point of X if, and only if, $x \in X$ and x is socially distanced from all points in X.

Observation L29.3: Let $X \subseteq \mathbb{R}$ and let $x \in \mathbb{R}$.
Then the following conditions are equivalent:

(a) x is a *limit point* of X.

(b) For all $\varepsilon > 0$ the set $X \cap [x - \varepsilon, x + \varepsilon]$ is *infinite.*

(c) There exists a sequence $(y_n)_{n=0}^{\infty}$ of elements of X that *is an injective function* such that $\lim_{n\to\infty} y_n = x$.

We will prove this observation in Module 29.

Note that (b) explains the meaning of the name "cluster point," and (c) explains the meaning of the name "limit point."

Note also that if we replace *"infinite"* by *"nonempty"* in (b) and if we *omit* the requirement that the sequence *be an injective function* in (c), we obtain properties that are equivalent to x being an *adherent point* of X.

The Heine-Borel Theorem for subsets of $\mathbb{R}$

Theorem 9.1.24: (Heine-Borel theorem for the line) Let X be a subset of $\mathbb{R}$. Then the following two statements are equivalent:

(a) X is closed and bounded.

(b) Given any sequence $(a_n)_{n=0}^{\infty}$ of real numbers which takes values in X (i.e., $a_n \in X$ for all n), there exists a subsequence $(a_{n_j})_{j=0}^{\infty}$ of the original sequence, which converges to some number L in X.

Proof: Assume X is bounded, and let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers which takes values in X.

Then this sequence is bounded, and by the Bolzano-Weierstrass theorem there exists a subsequence $(a_{n_j})_{j=0}^{\infty}$ of the original sequence that converges to some number $L \in \mathbb{R}$.

When X is closed, then L must be an element of X.

This proves the implication (a) $\Longrightarrow$ (b).

Theorem 9.1.24: Let $X \subseteq \mathbb{R}$. Then the following are equivalent:

(a) X is closed and bounded.

(b) Given any sequence $(a_n)_{n=0}^{\infty}$ of real numbers which takes values in X, there exists a subsequence $(a_{n_j})_{j=0}^{\infty}$ of the original sequence, which converges to some number L in X.

Proof: Now suppose that (a) fails. We distinguish two cases:

Case 1: X is unbounded.

Then we can recursively construct a sequence $(a_n)_{n=0}^{\infty}$ such that either $a_n \in (n, \infty)$ for all $n \in \mathbb{N}$ or $a_n \in (-\infty, -n)$ for all $n \in \mathbb{N}$. Such a sequence cannot have a convergent subsequence.

Case 2: X is not closed.

Then by the Corollary 9.1.17 there exists a convergent sequence $(a_n)_{n=0}^{\infty}$ of elements of X such that $L := \lim_{n\to\infty} a_n \notin X$. Then the limit of every subsequence of this sequence is also $L \notin X$.

In both cases (b) fails, which proves $\sim$(a) $\Longrightarrow$ $\sim$(b). $\square$

Take-home message

In this lecture we formally defined intervals in $\mathbb{R}$.

The empty set $\emptyset$ and singletons $\{a\} = [a, a]$ are *degenerate intervals.*

We also defined *bounded* sets of reals.

$x \in \mathbb{R}$ is an *adherent point* of $X \subseteq \mathbb{R}$ iff
$\forall \varepsilon > 0 \, \exists y \in X \ \ |x - y| \leq \varepsilon$, that is, iff
$\forall \varepsilon > 0 \ \ [x - \varepsilon, x + \varepsilon] \cap X \neq \emptyset$.

Moreover, $x \in \mathbb{R}$ is an adherent point of $X \subseteq \mathbb{R}$ iff there exists a sequence $(a_n)_{n=0}^{\infty}$, consisting entirely of elements of X, that converges to x.

The *closure* of $X \subseteq \mathbb{R}$, denoted $\overline{X}$, is the set of all the adherent points of X.

A set $X \subseteq \mathbb{R}$ is *closed* iff $X = \overline{X}$.

Take-home message, continued

Adherent points of a set $X \subseteq \mathbb{R}$ come in two distinct flavors.

Isolated points of X are elements of X such that there exists an interval $(x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$ that contains no points of X other than x itself.

Limit points aka *cluster points* of X are points in $\overline{X}$, not necessarily in X itself, such that for every $\varepsilon > 0$ the interval $(x - \varepsilon, x + \varepsilon)$ contains points of X other than x itself; infinitely many such points in fact.

$x \in \mathbb{R}$ is a limit point of $X \subseteq \mathbb{R}$ if, and only if, it is the limit of a sequence $(y_n)_{n=0}^{\infty}$ of elements of X that are all distinct.

The *Heine-Borel theorem for the line* asserts that a subset X of the real line $\mathbb{R}$ is closed and bounded if, and only if, every sequence of elements of X has a subsequence that converges to some $x \in X$.