Lecture 32: Continuous functions

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Companion to Advanced Calculus

Review: The definition of continuity

Let $X \subseteq \mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, and let $x_0 \in \mathbb{R}$ be an adherent point of X. We say that *$f(x)$ is continuous at x_0* if, and only if, the following three conditions are all satisfied:

1. $x_0 \in X$,
2. $\lim_{x \to x_0; x \in X} f(x)$ exists, and
3. $\lim_{x \to x_0; x \in X} f(x) = f(x_0)$.

The third of these conditions is actually redundant as in this version it follows from the first two.

When x_0 is a limit point of X, then we can replace the second of the above conditions by the requirement that

$\lim_{x \to x_0} f(x) := \lim_{x \to x_0; x \in X \setminus \{x_0\}} f(x)$ exists.

This version of the definition is then no longer redundant and is a rigorous analogue of the familiar definition from calculus.

Polynomial functions and rational functions are continuous

Let $X \subseteq \mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, and let $x_0 \in \mathbb{R}$ be an adherent point of X.

- When f is *not* continuous at x_0, then we say that *is discontinuous at x_0* or *has a discontinuity at x_0.*
- When $f(x)$ is continuous at every $x_0 \in X$, that is, for every x_0 in the domain of this function, then we say that $f(x)$ *is continuous on X* or simply *$f(x)$ is continuous.*

As Corollary 31.2 shows, all polynomial functions, and more generally, all rational functions, that is functions $\frac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomial functions, are continuous.

While they are undefined for x where $q(x) = 0$, they are continuous at all x_0 in their domains.

Review: Two examples from Conversation 13

Example L32.1: Let $X \subseteq \mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, and let $x_0 \in X$.

When x_0 is an isolated point of X, then f is continuous at x_0.

Example L32.2: The characteristic function $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ that takes the value $f(x) = 1$ for all $x \in \mathbb{Q}$ and the value $f(x) = 0$ for all $x \in \mathbb{R}\backslash\{\mathbb{Q}\}$ is an example of a function that is discontinuous at every $x_0 \in \mathbb{R}$.

This follows from the fact that for each $x_0 \in \mathbb{R}$ we have:

$\lim_{x\to x_0; x\in\mathbb{Q}\backslash\{x_0\}} \chi_{\mathbb{Q}}(x) = 1$ and

$\lim_{x\to x_0; x\in\mathbb{R}\backslash(\mathbb{Q}\cup\{x_0\})} \chi_{\mathbb{Q}}(x) = 0$,

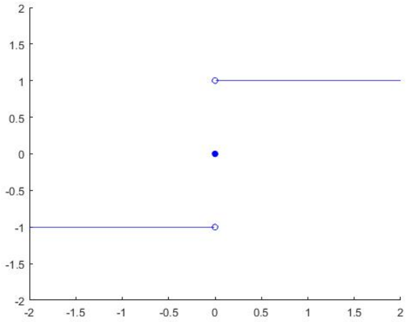
so that $\lim_{x\to x_0; x\in\mathbb{R}} \chi_{\mathbb{Q}}(x)$ does not exist.

One more example

Example L32.3: The *signum function* $sgn : \mathbb{R} \to \mathbb{R}$, defined as

- $sgn(x) = 1$ when $x > 0$,
- $sgn(x) = 0$ when $x = 0$,
- $sgn(x) = -1$ when $x < 0$,

has a *jump discontinuity* at $x_0 = 0$ and is continuous at all $x_0 \neq 0$.



Proposition 9.4.7: (Equivalent formulations of continuity)
Let $x_0 \in X \subseteq \mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function.
Then the following statements are logically equivalent:

(a) f is continuous at x_0.

(b) For every sequence $(a_n)_{n=0}^{\infty}$ consisting of elements of X with $\lim_{n\to\infty} a_n = x_0$, we have $\lim_{n\to\infty} f(a_n) = f(x_0)$.

(c) $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \; (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon)$.

(d) $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \; (|x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \varepsilon)$.

Note that Proposition 9.3.9 implies that $\lim_{x\to x_0; x\in X} f(x) = f(x_0)$ if, and only if, for every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements of X and converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to $f(x_0)$.

The equivalence between (a) and (b) then follows from the definition of continuity at x_0.

The equivalence between (a), (d), and (c)

Proposition 9.4.7: (Equivalent formulations of continuity)
Let $x_0 \in X \subseteq \mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function.
Then the following statements are logically equivalent:

(a) f is continuous at x_0.

(b) For every sequence $(a_n)_{n=0}^{\infty}$ consisting of elements of X with $\lim_{n\to\infty} a_n = x_0$, we have $\lim_{n\to\infty} f(a_n) = f(x_0)$.

(c) $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \ \ (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon)$.

(d) $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \ \ (|x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \varepsilon)$.

The definition of continuity at x_0 directly translates into (d):
$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \ \ (|x - x_0| < \delta \implies |f(x) - f(x_0)| \leq \varepsilon)$,
which is implied by (c), since for all $x \in X$
$|f(x) - f(x_0)| < \varepsilon \implies |f(x) - f(x_0)| \leq \varepsilon$.

On the other hand, if $f(x)$ is continuous at x_0, then for any given $\varepsilon > 0$ we can find $\delta > 0$ such that for all $x \in X$
$|x - x_0| < \delta \implies |f(x) - f(x_0)| \leq \frac{\varepsilon}{2} < \varepsilon$, which gives (c).

Other definitions of continuity of a function

It follows from the results on the previous two slides that we can characterize continuity of a function as follows:

Proposition L32.1: (Equivalent formulations of continuity)
Let $X \subseteq \mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function.
Then the following statements are logically equivalent:

(a) f is continuous.

(b) For every sequence $(a_n)_{n=0}^{\infty}$ consisting of elements of X with $\lim_{n\to\infty} a_n = L \in X$, we have $\lim_{n\to\infty} f(a_n) = f(L)$.

(c) $\forall x_0 \in X \, \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \; (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon)$.

(d) $\forall x_0 \in X \, \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in X \; (|x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \varepsilon)$.

Question L32.1: Is the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) := \frac{1}{x}$ continuous?

Yes. In calculus you probably learned that this function has an "infinite discontinuity" at $x_0 = 0$. But since 0 is not in the domain of f, in the sense of our definition, the function f would still be considered a continuous function.

The function $\sin x$ is also continuous

Trig functions have not been formally defined in the textbook yet, but we can somewhat informally use our geometric intuitions.

In particular, we can convince ourselves by drawing pictures that $\lim_{x\to 0} \sin x = \sin 0 = 0$ and $\lim_{x\to 0} \cos x = \cos 0 = 1$.

Proposition L32.2: The function $\sin x$ is continuous.

Proof: We need to show that $\forall x_0 \in \mathbb{R} \quad \lim_{x\to x_0} \sin x = \sin x_0$.

Let $x_0 \in \mathbb{R}$. Also, let $y := x - x_0$.

Question L32.2: How can we express $\lim_{x\to x_0} \sin x$ in terms of y?

$\lim_{x\to x_0} \sin x = \lim_{y\to 0} \sin(x_0 + y)$

$= \lim_{y\to 0}(\sin x_0 \cos y + \cos x_0 \sin y)$

$= \lim_{y\to 0} \sin x_0 \cos y + \lim_{y\to 0} \cos x_0 \sin y$

$= \sin x_0 \lim_{y\to 0} \cos y + \cos x_0 \lim_{y\to 0} \sin y$

$= 1 \sin x_0 + 0 \cos x_0 = \sin x_0. \quad \square$

Note that the last step of the argument on the previous slide used part (d) of the following result:

Proposition 9.3.14: (Limit laws for functions) Let $E \subseteq X \subseteq \mathbb{R}$, let x_0 be an adherent point of E, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions, and let $c \in \mathbb{R}$.

If $\lim_{x \to x_0; x \in E} f(x) = L$ and $\lim_{x \to x_0; x \in E} g(x) = M$, then:

(a) $\lim_{x \to x_0; x \in E} (f + g)(x) = L + M$,

(b) $\lim_{x \to x_0; x \in E} (f - g)(x) = L - M$,

(c) $\lim_{x \to x_0; x \in E} (fg)(x) = LM$,

(d) $\lim_{x \to x_0; x \in E} (cf)(x) = cL$,

(e) $\lim_{x \to x_0; x \in E} (f/g)(x) = L/M$, provided $M \neq 0$,

(f) $\lim_{x \to x_0; x \in E} \min(f, g)(x) = \min(L, M)$,

(g) $\lim_{x \to x_0; x \in E} \max(f, g)(x) = \max(L, M)$.

Some consequences of Proposition 9.3.14

Proposition 9.4.9: (Arithmetic preserves continuity)
If $f, g : X \to \mathbb{R}$ are continuous functions, then the functions $f + g, f - g, fg, f/g, \min(f, g), \max(f, g)$ are also continuous.

Any linear combination $c_1 f_1 + \cdots + c_n f_n$ of continuous functions $f_1, \ldots, f_n$ with a common domain X is a continuous function.

These results also apply to continuity at a given x_0.

We will show shortly that $\cos x$ is also continuous.
Thus each of the following functions is continuous:

- $\tan x := \frac{\sin x}{\cos x}$,
- $\cot x := \frac{\cos x}{\sin x}$,
- $\sec x := \frac{1}{\cos x}$,
- $\csc x := \frac{1}{\sin x}$.
- All linear combinations of trig functions, like, for example $f(x) := 3 \tan x - 45 \csc x$.

More examples of continuous functions

Proposition 9.4.12: (Absolute value is continuous)
The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := |x|$ is continuous.

Proof: Note that $|x| = \max(-x, x)$.

Thus the result follows from Proposition 9.4.9. $\square$

The textbook also gives the following results:

Proposition 9.4.10: For every positive real number $a > 0$ the exponential function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := a^x$ is continuous.

Proposition 9.4.11: Let p be a real number. Then the power function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) := x^p$ is continuous.

Compositions of continuous functions are continuous

Proposition 9.4.13: (Composition preserves continuity) Let $X, Y \subseteq \mathbb{R}$, and let $f : X \to Y$ and $g : Y \to \mathbb{R}$ be functions. Let x_0 be a point in X. If f is continuous at x_0, and g is continuous at $f(x_0)$, then the composition $g \circ f : X \to \mathbb{R}$ is continuous at x_0.

In particular, if both f and g are continuous, so is $g \circ f$.

Proof: Let X, Y, f, g, x_0 be as in the assumptions.

Let $(a_n)_{n=0}^{\infty}$ be a sequence consisting of elements of X with $\lim_{n\to\infty} a_n = x_0$.

Then by the assumptions on f, the sequence $(f(a_n))_{n=0}^{\infty}$ consists of elements of Y and we have $\lim_{n\to\infty} f(a_n) = f(x_0)$.

Moreover, by the assumptions on g, for the sequence $(g(f(a_n)))_{n=0}^{\infty}$ we have $\lim_{n\to\infty} g(f(a_n)) = g(f(x_0))$.

This means that $\lim_{n\to\infty}(g \circ f)(a_n) = (g \circ f)(x_0)$.

Now the result follows from Proposition 9.4.7. $\square$

More examples of continuous functions

Let us illustrate with a few examples how the results of this lecture allow us to determine that certain functions are continuous.

- We have shown that $\sin x$ is continuous. $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ for all $x \in \mathbb{R}$. Thus $\cos = \sin \circ f$, where $f(x) := x + \frac{\pi}{2}$ is a polynomial function. Now it follows from Proposition 9.4.13 that cos is a continuous function.
- It now follows from Propositions 9.4.12 and 9.4.13 that $|\cos x|$ is also a continuous function.
- By Propositions 9.4.10 and 9.4.11, each of the following functions is also continuous on the interval $[0, \infty)$: $f(x) := e^x$ and $g(x) = \sqrt{x}$.
- It now follows from Propositions 9.4.13 and 9.4.9 that the function $e^x - \cos\sqrt{x}$ is also continuous on $[0, \infty)$.

Take-home message

In this lecture we proved continuity for a large class of functions. We now know that the following types of functions are continuous, that is, continuous at every point in their domains:

- Polynomial functions.
- Rational functions.
- Exponential functions a^x.
- Power functions x^p.
- The absolute value function.
- All trig functions: $\sin x, \cos x, \tan x, \cot x, \sec x, \csc x$.
- Sums, differences, products, and quotients of continuous functions.
- Minima and maxima of continuous functions.
- Compositions of continuous functions.